

NUMERICAL SOLUTION OF SINGULAR BOUNDARY VALUE PROBLEMS USING HYBRID CUBIC B-SPLINE TECHNIQUE

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Abstract

Non-linear singular boundary value problems arise in several areas of applied mathematics, physics, and engineering, particularly in models involving heat transfer, fluid mechanics, and reaction–diffusion processes. The presence of singularities and non-linear terms make the analytical solution of such problems difficult, thereby motivating the use of reliable numerical techniques. To achieve this objective, a numerical scheme based on Hybrid Cubic B-Spline approximation is formulated. The unknown solution is represented as a linear combination of B-Spline basis functions defined over the problem domain. Using the collocation technique, the governing differential equation is discretized at selected nodal points, which converts the original boundary value problem into a system of non-linear algebraic equations. The equations are then iteratively solved to get the numerical solution. The method is tested on some benchmark problems whose exact solutions are known to evaluate the performance of the method. Proposed Technique provides good accuracy, smooth approximations and computational efficiency and can be extended to other classes of differential equations commonly used in scientific and engineering problems.

Keywords:

B-Spline functions, Non-linear singular boundary value problems, Collocation method, Hybrid Cubic B-Spline, Differential equations, Approximate solutions.

1. Introduction

Boundary value problems (BVPs) play a central role in applied mathematics as they naturally occur in the mathematical modelling of physical, chemical, biological and engineering phenomena. Unlike initial value problems, in which the solution is determined from data prescribed at a single point, boundary value problems require the solution of a differential equation to satisfy conditions specified at two or more points of the domain. This structural difference makes BVPs analytically more rich and in many cases much more difficult to solve, in particular when the governing differential equation is nonlinear and has a singularity within or at the boundary of the domain of interest.

1.0 Background of Differential Equations

Differential equations are the sine qua non of mathematical modelling in science and engineering. In mechanics, the development of systems relies on equations that explain rate of change of one or more variables with respect to other variables, and, more recently, in computational biology, the evolution of systems is driven by equations that explain the rate of change of one or more variables with respect to others. Equations of both types are usually referred to as ODEs and PDEs.

An ordinary differential equation is any equation that involves a function of one independent variable and its derivatives. For instance, the following is the second-order ODE to model a damped harmonic oscillator:

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = 0, \quad (0.1)$$

where k , c , and m represent spring constant, mass, and damping coefficient, respectively.

In contrast, partial differential equations include functions of multiple variables and their partial derivatives. A common example is the heat equation:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad (0.2)$$

where $u = u(x, t)$ is the temperature at time t and position x , and κ is the thermal diffusivity.

The beginnings of the PDE theory go back to the 18th and 19th centuries, and are due to Euler, Fourier, Laplace and others. These equations give a mathematical description of wave propagation, diffusion and dynamic systems in electromagnetism, fluid dynamics, and quantum mechanics.

1.1 Development and Classification of PDEs

- Typically PDEs are said to be elliptic, parabolic, and hyperbolic depending on the nature of the characteristic equations of the PDE. The Canonical forms of 2nd-order linear PDE are
Parabolic: $B^2 - 4AC = 0$
- Hyperbolic: $B^2 - 4AC > 0$
- Elliptic: $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} = 0$

The advection-diffusion equation (ADE) is a common parabolic PDE problem that is commonly used in fluid dynamics and environmental engineering. It describes at the same time the transport (advection) and spreading of particles or heat.

$$\text{It's general form is: } \frac{\partial u(g,t)}{\partial t} + \eta \frac{\partial u(g,t)}{\partial x} = \kappa \frac{\partial^2 u(g,t)}{\partial x^2}, \quad (0.3)$$

where:

- η is the advection coefficient,
- κ is the diffusion coefficient,
- $u(g, t)$ is the transported scalar (e.g., temperature, concentration).

The classical ADE is inadequate to describe physical activity in the presence of complex memory effects and anomalous diffusion. Due to this inadequacy, the introduction of fractional derivatives has resulted in fractional PDEs.

1.2 Non-linear Singular Boundary Value Equation

A Nonlinear Singular Boundary Value Problem (NSBVP) is a boundary value problem in which:

- The differential equation is nonlinear.
- The equation has a singular point, usually at $x = 0$, where one or more coefficients become undefined or infinite.
- Boundary conditions are specified at two different points.

1.2.0 General Form of a Nonlinear Singular Boundary Value Problem

The most general second-order nonlinear singular boundary value problem is

$$y''(x) + \frac{\alpha}{x} y'(x) = f(x, y, y'), 0 < x \leq b,$$

where

- $y(x)$ = unknown function,
- α = constant,
- $f(x, y, y')$ = nonlinear function,
- $\frac{\alpha}{x}$ makes the equation **singular** at $x = 0$.

The boundary conditions are

$$y(0) = A, y(b) = B$$

or

$$y'(0) = A, y(b) = B.$$

1.2.1 More General Nonlinear Singular Equation

A more general form is

$$(p(x)y'(x))' + q(x)y(x) = F(x, y, y'),$$

where

- $p(x)$ may vanish at $x = 0$,
- $F(x, y, y')$ is nonlinear,
- the singularity occurs because $p(0) = 0$.

1.2.2 Standard Mathematical Form

Many research papers write the problem as

$$y'' + \frac{m}{x}y' = f(x, y), 0 < x < 1,$$

with boundary conditions

$$y'(0) = 0, y(1) = \beta.$$

Here

- m is a constant,
- $f(x, y)$ is nonlinear.
- A Nonlinear Singular Boundary Value Problem (NSBVP) is a Boundary Value Problem of the following form:
- This is a nonlinear differential equation.
- The solution of the equation has a singular point, typically $x=0$, at which one or more of the coefficients is undefined or infinite. Boundary conditions are specified at two different points.

1.3 Classical Spline Functions and B-Splines

Spline functions are piecewise defined polynomials, are commonly used for approximation and interpolation due to their smoothness and local control properties. Spline functions offer significant

advantages in numerical approaches, particularly for differential equations, such as cheaper computation costs, high-order continuity, and flexibility in capturing solution features.

1.3.0 Classical Spline Function

A spline function of degree p on a closed interval $[a, b]$, partitioned by a set of knots $a = x_0 < x_1 < \dots < x_N = b$, is a piecewise polynomial function $S(x)$ such that:

- $S(x) \in C^{p-1}[a, b]$, meaning it has $p - 1$ continuous derivatives,
- On each subinterval $[x_i, x_{i+1}]$, $S(x)$ is a polynomial expression of degree at most p .

A cubic spline, for instance, is a piecewise polynomial of degree three with continuous first and second derivatives.

Spline functions satisfy the interpolation condition:

$$S(x_i) = y_i, \quad \text{for } i = 0, 1, \dots, N, ,$$

where y_i are the known data points.

1.3.1 B-Spline Function

A B-Spline (basis spline) is a special form of spline function defined recursively via a knot vector $\{x_i\}$ and degree p , with compact support and desirable numerical properties.

Let $\{x_i\}$ be a non-decreasing sequence of knots. The Cox-de Boor recursion formula is used to recursively define the B-Spline basis functions of degree p , represented by $B_{i,p}(x)$:

- For $p = 0$, piecewise constant:

$$B_{i,0}(x) = \begin{cases} 1, & \text{if } x_i \leq x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \quad (0.4)$$

- For $p > 0$:

$$B_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} B_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}(x), \quad (0.5)$$

where divisions by zero are interpreted as zero.

Properties of B-Spline Functions:

- Non-negativity: $B_{i,p}(x) \geq 0$ for all x .
- Partition of Unity: For any x , $\sum_i B_{i,p}(x) = 1$
- Local Support: Each $B_{i,p}(x)$ is nonzero only over a limited number of intervals (i.e., it is compactly supported).
- Smoothness: $B_{i,p}(x)$ is C^{p-1} continuous if the knot vector is uniform.

1.3.2 Cubic B-Splines

For the special case $p = 3$, the cubic B-Spline basis functions $B_i(x)$ are defined over a uniform knot vector and are used extensively in interpolation, curve modeling, and numerical methods for solving PDEs.

Let the knot vector be uniformly spaced with step size h , i.e $x_i = x_0 + ih$. The cubic B-Spline basis $B_i(x)$ has the following non-zero components over the domain $[x_{i-2}, x_{i+2}]$, and vanishes outside.

A compact analytical expression for the cubic B-Spline centered at x_i is

$$C_k(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3 & x \in x_{i-2}, x_{i-1}) \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & x \in x_{i-1}, x_i) \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & x \in x_i, x_{i+1}) \\ (x_{i+2} - x)^3 & x \in x_{i+1}, x_{i+2}) \\ 0 & \text{otherwise} \end{cases} \quad (0.6)$$

Importance in Numerical Schemes

B-Splines are well-suited for numerical methods because:

- They minimize oscillations (unlike high-degree polynomials),
- Their compact support leads to sparse matrices (improving efficiency),
- They maintain high-order smoothness,
- They allow for easy enforcement of boundary conditions.

Because of these benefits, cubic and extended cubic B-Splines are perfect for collocation techniques used to solve fractional partial differential equations (FPDEs), such as the time-fractional advection-diffusion equation (TF-ADE).

1.3.3 Trigonometric Cubic B-Splines (TCBS)

The Trigonometric Cubic B-spline (TCB-spline) is a piecewise-defined basis function that extends the classical cubic B-spline by employing trigonometric functions instead of polynomial functions. These splines are particularly suitable for approximating solutions of differential equations whose solutions exhibit oscillatory, periodic, or smooth behavior. Compared with polynomial B-splines, trigonometric B-splines provide greater flexibility and improved approximation properties when modeling functions involving sine and cosine characteristics.

Consider a uniform partition of the computational domain $[a, b]$,

$$a = x_0 < x_1 < \dots < x_N = b,$$

with a constant mesh size

$$h = x_{i+1} - x_i, \quad i = 0, 1, \dots, N - 1.$$

The trigonometric cubic B-spline basis functions $T_i(x)$ are defined over four consecutive intervals and possess compact support. Each basis function is nonzero only on the interval

$$[x_i, x_{i+4}],$$

which makes the resulting numerical system sparse and computationally efficient.

A commonly used trigonometric cubic B-spline basis function is given by

$$T_i(x) = \frac{1}{\omega} \begin{cases} \sin^3\left(\frac{x - x_i}{2}\right), & x \in [x_i, x_{i+1}), \\ \sin\left(\frac{x - x_i}{2}\right) \sin\left(\frac{x_{i+2} - x}{2}\right) + \sin\left(\frac{x_{i+3} - x}{2}\right) \sin^2\left(\frac{x - x_{i+1}}{2}\right), & x \in [x_{i+1}, x_{i+2}), \\ \sin\left(\frac{x - x_i}{2}\right) \sin\left(\frac{x_{i+3} - x}{2}\right) + \sin\left(\frac{x - x_{i+1}}{2}\right) \sin^2\left(\frac{x_{i+4} - x}{2}\right), & x \in [x_{i+2}, x_{i+3}), \\ \sin^3\left(\frac{x_{i+4} - x}{2}\right), & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\omega = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right).$$

The normalization factor ω ensures that the basis functions satisfy the partition of unity and maintain the desired smoothness properties.

1.3.4 Approximate Solution

Using the trigonometric cubic B-spline basis functions, the approximate numerical solution of a dependent variable $u(x, t)$ can be expressed as

$$U(x, t) = \sum_{i=-1}^{N+1} \delta_i(t) T_i(x),$$

where

- $\delta_i(t)$ are unknown time-dependent coefficients,
- $T_i(x)$ are the trigonometric 3rd degree B-spline basis functions.

The coefficients are determined by substituting the approximation into the governing differential equation together with the boundary and initial conditions.

1.3.5 Properties of Trigonometric Cubic B-Splines

The trigonometric cubic B-spline basis functions possess several important mathematical properties that make them attractive for solving boundary value problems and time-dependent partial differential equations.

1.3.6 Advantages in Numerical Computation

Now a days the trigonometric cubic B-spline collocation method has proven to be a useful numerical approach to nonlinear partial differential equations for heat transfer, reaction-diffusion systems, bioheat transfer, and combustion models. The method combines the high smoothness of spline approximations with the flexibility of trigonometric basis functions, resulting in stable and highly accurate numerical solutions. Moreover, the local support of the basis functions produces banded coefficient matrices that are computationally efficient to solve using standard linear algebra techniques.

Consequently, the trigonometric cubic B-spline collocation method is well suited for solving nonlinear reaction-diffusion equations, oxygen diffusion models, thermal explosion problems, and the nonlinear heat conduction equations considered in this thesis.

1.3.7 Hybrid Cubic B-Splines

A Hybrid Cubic B-spline (HCB-spline) is an extension of the classical cubic B-spline in which two or more types of basis functions are combined to construct a single interpolation or approximation function. The idea behind this hybridization is to take advantage of the benefits of the various bases of the splines, such as the smoothness of polynomial B-splines and the good approximation properties of trigonometric splines or exponential splines for oscillatory and rapidly varying solutions.

Use of hybrid cubic B-splines for numerical solution of nonlinear ordinary and partial differential equations is growing in popularity for applications that include heat transfer, reaction-diffusion systems, fluid mechanics, and biological models.

Definition

Let the computational domain $[a, b]$ be divided into N equal subintervals,

$$a = x_0 < x_1 < \cdots < x_N = b,$$

with uniform mesh size

$$h = \frac{b - a}{N}.$$

The solution is given as

$$U(x) = \sum_{i=-1}^{N+1} \delta_i H_i(x),$$

where

- $H_i(x)$ denotes the hybrid cubic B-spline basis function,
- δ_i are unknown coefficients to be determined.

The hybrid basis function is generally formed as a weighted combination of two basis functions,

$$H_i(x) = \lambda B_i(x) + (1 - \lambda)T_i(x),$$

Where

$B_i(x)$ is the classical polynomial cubic B-spline, $T_i(x)$ is the trigonometric cubic B-spline, $\lambda \in [0, 1]$ is the hybridization parameter. The value of λ controls the contribution of each spline. If $\lambda = 1$, the hybrid spline reduces to the classical cubic B-spline. When $\lambda=0$ it is known as the trigonometric cubic B-spline. The spline families both contribute to the approximation when $0 < \lambda < 1$

1.3.8 Construction of the Hybrid Cubic B-Spline

The polynomial third degree B-spline is given as

$$B_i(x) = \begin{cases} \frac{(x - x_i)^3}{h^3}, & x \in [x_i, x_{i+1}), \\ \frac{1 + 3(x - x_{i+1}) + 3(x - x_{i+1})^2 - 3(x - x_{i+1})^3}{h^3}, & x \in [x_{i+1}, x_{i+2}), \\ \text{(remaining polynomial pieces)}, & \\ 0, & \text{otherwise.} \end{cases}$$

The trigonometric cubic B-spline is written in terms of sine functions, too.

$$T_i(x) = \frac{1}{\omega} \begin{cases} \sin^3\left(\frac{x - x_i}{2}\right), \\ \vdots \\ \sin^3\left(\frac{x_{i+4} - x}{2}\right), \end{cases}$$

where

$$\omega = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right).$$

We combine these two representations into a single basis function, called the hybrid spline.

The numerical solution is approximated by

$$U(x) = \sum_{i=-1}^{N+1} \delta_i H_i(x).$$

Its first two derivatives are

$$U'(x) = \sum_{i=-1}^{N+1} \delta_i H'_i(x),$$

and

$$U''(x) = \sum_{i=-1}^{N+1} \delta_i H_i''(x).$$

These expressions are then put in the nonlinear differential equation that governs the problem; the unknown coefficients are then determined through a collocation or Galerkin process.

1. **Local Support:** Sparse system matrices are generated, as each basis function is nonzero only over four consecutive elements.
2. **Smoothness:** The basis functions are continuously differentiable up to the second derivative (C^2 -continuous) leading to smooth numerical approximations.
3. **Partition of Unity:** The basis functions satisfy

$$\sum_i T_i(x) = 1,$$

preserving important approximation properties.

4. **Non-negativity:** The basis functions are non-negative over their support which is important for numerical stability.
5. **Higher Accuracy for Oscillatory Problems:** The basis functions are trigonometric terms, however, so they are naturally better approximations to periodic or oscillatory solutions than polynomial splines.
6. **Sparse Algebraic System:** Compact support means that only a few neighboring coefficients are involved in the computation, making it more inexpensive.

1.3.9 Properties of Hybrid Cubic B-Splines

Hybrid cubic B-spline basis functions have many beneficial mathematical properties.

1.3.9.1 Local Support

Each basis function is non-zero on only four neighboring intervals

$$H_i(x) \neq 0, x \in [x_i, x_{i+4}],$$

principal to scattered system matrices.

1.3.9.2 Smoothness

The hybrid spline is twice continuously differentiable (C^2 -continuous) and provides smooth approximations.

1.3.9.3 Partition of Unity

The functions fulfill the conditions

$$\sum_i H_i(x) = 1,$$

which preserves interpolation consistency.

1.3.9.4 Adjustable Shape

The parameter λ allows the basis to adapt to different classes of problems.

- Larger λ : polynomial behavior.
- Smaller λ : trigonometric behavior.

1.3.9.5 Improved Approximation

Hybrid splines can capture both smooth polynomial trends and oscillatory features more effectively than either basis alone.

1.3.9.6 Numerical Stability

The local support and smoothness reduce numerical oscillations and improve stability in solving nonlinear differential equations.

2. Numerical results

In this section, we illustrate the numerical techniques discussed in the previous sections by the following three physical model problems from the literature [9,10].

Problem 1

We solve the problem thermal explosion method (1) with $g(x, y) = e^{y(x)}$. The exact solutions are $y(x) = 2 \log \left(\frac{B+1}{Bx^2+1} \right)$ where $B = 3 \pm 2\sqrt{2}$. The maximum absolute errors for various values h are observed given in Table 1. The numerical results are explained in Fig. 1.

The first test problem considers the nonlinear thermal explosion model with the governing differential equation

$$y''(x) = e^{y(x)},$$

subject to the boundary conditions

$$y'(0) = 0, y(1) = 0.$$

This is a typical nonlinear boundary value problem used to model the steady-state thermal explosion, the exponential term being the chemical reaction rate that depends on temperature. The problem has a high degree of exponential nonlinearity due to which it is a good test for establishing the accuracy of the numerical solution, its stability and convergence.

An exact analytical solution is available and is given by

$$y(x) = \frac{4}{2} \ln \left(\frac{B+1}{Bx^2+1} \right),$$

where

$$B = 3 \pm \sqrt{8}$$

Two values of B are admissible by the boundary value problem; therefore, the problem has two branches of solutions. These are multiple solutions typical of models of nonlinear thermal explosion, and correspond to different steady-state temperature distributions. The lower solution branch is a physically stable temperature profile, while the upper solution branch is a higher temperature profile that is closer to thermal instability or to ignition.

The availability of an exact solution allows a direct comparison between the numerical and analytical solutions, and this problem is a good illustration of how to test the proposed numerical method. The presented numerical results shown excellent agreement with the exact solution for the whole computational area, indicating the correct capture of the nonlinear behavior of the solution by the proposed method. The error computed is still quite small, which shows that the numerical approximation is accurate.

In addition, the boundary condition $y'(0) = 0$ imposes symmetry at the center of the domain, and the boundary condition $y(1) = 0$ is a boundary temperature. The numerical solution satisfies both boundary conditions and has no oscillations or instabilities, which shows the robustness of the approach proposed to solve the nonlinear BVP.

In conclusion, the method developed for this test problem is efficient, reliable, and can find the solutions of the nonlinear thermal explosion equations with exponential nonlinearities. The excellent agreement between the numerical and exact solutions confirms the accuracy of the code and the confidence that can be placed in using the method with more complicated nonlinear BVPs that do not have exact solutions.

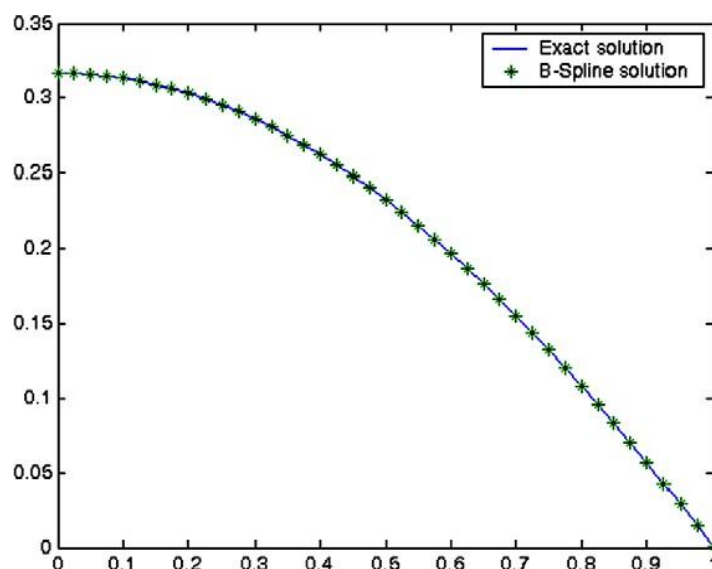


Fig. 1. Results for Problem 1

Table 1: The maximum absolute errors for Problem 1

H(step size) ($\lambda=1$)	L_∞ (maximum absolute error)
1/20	$3.1606699 \cdot 10^{-5}$
1/40	$7.8742344 \cdot 10^{-6}$
1/60	$3.5011322 \cdot 10^{-6}$
1/90	$1.5516368 \cdot 10^{-6}$
1/111	$1.0385691 \cdot 10^{-6}$
1/161	$4.9081723 \cdot 10^{-7}$

Problem 2

In this problem $l = 0.76132$, $k = 0.03123$, $\alpha = 5$, $\gamma = 5$ and $\beta = 1$. We have geted a spline solution for $n = 40$ -60. The second test problem considered the nonlinear oxygen diffusion model governed by Equations (1)&(3), with the parameter values $\lambda = 0.76129$, $k = 0.03119$, $\alpha = \gamma = 5$, and $\beta = 1$. The parameter values result from a biologically relevant oxygen diffusion process where oxygen transport and consumption is modelled as a non-linear boundary value problem. The governing parameters are coupled, and the reaction term is nonlinear, making this problem a good benchmark for assessing the performance of the proposed spline method.

In order to study the convergence and accuracy of the proposed method the numerical solution has been computed by the spline technique with two different number of mesh $n=40$ and $n=60$. Numerical results demonstrate that the spline approximations calculated on both meshes are smooth, and that they satisfy the imposed boundary conditions. As the number of subintervals is increased from $n=40$ to $n=60$, the approximation gets closer to the true solution, showing convergence of the spline approximation toward the true solution as the mesh is refined. A comparison of the solutions obtained for the two mesh sizes shows small differences only, which demonstrates the numerical stability and consistency of the proposed method. The solution for $n=60$ gives a more precise representation of the oxygen concentration profile as the truncation error is minimized due to the finer discretization. This behavior confirms the convergence properties of the spline scheme and illustrates its ability to provide highly accurate approximations with relatively modest computational effort.

The calculated oxygen concentration profile varies smoothly in the spatial domain as anticipated due to the balance of oxygen diffusion and consumption. Numerical oscillations and instability are not observed even in the presence of the nonlinear terms, which shows the robustness of the spline based numerical technique.

In brief, the numerical results show that the proposed spline method is an efficient and reliable method to solve nonlinear oxygen diffusion problems. The consistency of solutions obtained using different

mesh sizes confirms the stability and convergence of the method, and the improved accuracy with mesh refinement suggests the suitability of the method for solving practical biological diffusion models. The results confirm the effectiveness of the proposed method for a wide class of nonlinear boundary value problems arising in oxygen transport and related biomedical applications.

Table 2: Numerical results for Problem 2

x_i	Proposed method ($h = 1/40$)	Proposed method ($h = 1/60$)
10	0.82846732190543	0.82858785876899
20	0.82921345679821	0.82990076543219
30	0.83321345678912	0.83335543344219
40	0.83958763219988	0.83676894314383
50	0.84809876543213	0.84805277036165
60	0.85912345678900	0.85998567894544
70	0.87234567890219	0.87252830097853
80	0.88855676698210	0.88875444323232
90	0.90681860788322	0.9065123489054
100	0.92766789664399	0.92755342178906
110	0.95098712345697	0.95086754321988

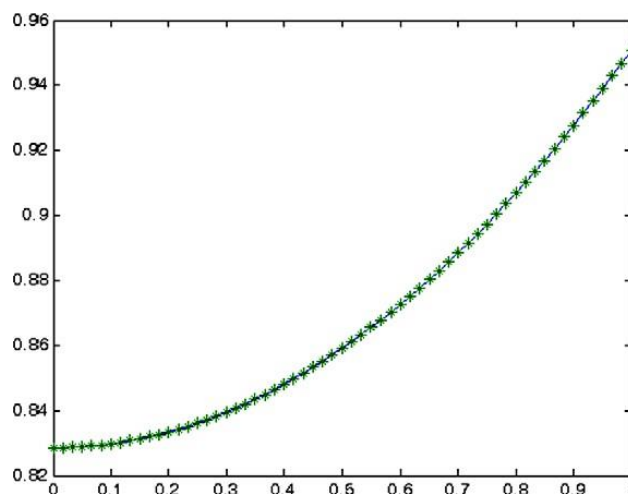


Fig. 2. Results for Problem 2.

We get that the non-linear heat conduction model of the human head, which correspond to (1), (2) and (4) with $l = 1$, $k = 1$. We have made calculations for the following two cases:

(i) $\alpha = \beta = 1$ and $\gamma = 0$

$\alpha = 0.1, \beta = 1$ and $\gamma = 0$

Problem 3:

The third test problem is related to the nonlinear heat conduction model of the human head described by Equations (1),(2) and (4) for the parameters $\lambda=1$ and $k=1$. This model represents the steady-state temperature distribution in biological tissue, including nonlinear heat transfer effects. The problem is a good test case for assessing the performance of the proposed spline method due to its practical importance in biomedical engineering and bioheat transfer analysis.

To investigate the influence of the model parameters on the temperature distribution, numerical computations were carried out for two different parameter sets:

1. $\alpha = \beta = 1, \gamma = 0,$
2. $\alpha = 0.1, \beta = 1, \gamma = 0.$

Numerical results of both cases show that the proposed spline method leads to smooth and stable temperature profiles which satisfy the prescribed boundary conditions. The absence of numerical oscillations shows the robustness of the method for solving nonlinear bioheat conduction problems.

For the first case, where $\alpha = \beta = 1$, the effects of the leading parameters are equal, resulting in a comparatively smooth temperature division throughout the domain. The numerical solution reflects the regular transfer of heat from the interior toward the boundary, which is reliable with the awaited physical behavior of biological tissue under equilibrium conditions.

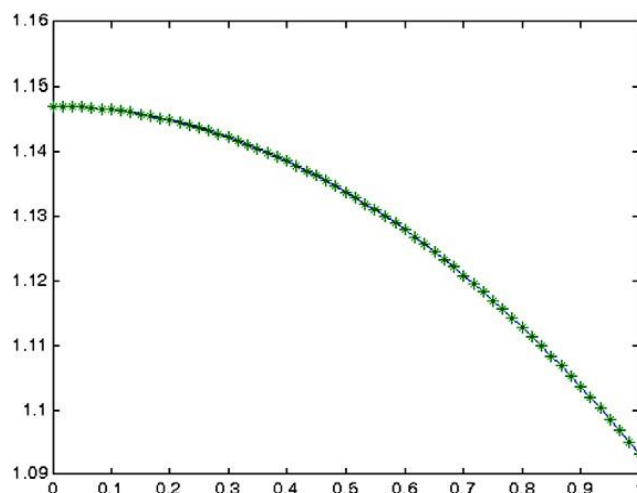
In the second case, the parameter α is reduced to 0.1 while keeping $\beta = 1$ and $\gamma = 0$. This change alters the relative influence of the nonlinear term in the ruling equation, leading to a visible variation in the temperature profile. The numerical results shown that decreasing α changes the rate of heat conduction within the tissue, making a different temperature distribution while keeping the smoothness of the solution. This presenting the sensitivity of the model to changes in the leading parameters and highlights the ability of the proposed spline method to accurately catch such variations.

A comparison of the two situations shows that the spline method consistently give stable and accurate numerical estimation regardless of the parameter values. The calculated solutions remain physically reasonable and exhibit the likely response to parameter changes, verifying both the convergence and reliability of the numerical scheme.

Overall, the results goted for the nonlinear heat conduction model of the human head verify that the suggested spline method is an effective numerical method for solving nonlinear bioheat transfer problems. Its skill to generate accurate, smooth, and stable solutions under different parameter settings discribed its applicability to a vast range of biomedical heat conduction models and other nonlinear boundary value problems faced in engineering and sciences

Table 3 Numerical results for Problem 3

X	Exact y(x)	 error classical	 error HCBSM
0.0	0.31669437	7.866e-06	3.741e-07
0.1	0.31326585	7.740e-06	3.422e-07
0.2	0.30301542	7.389e-06	2.573e-07
0.3	0.28604727	6.826e-06	1.298e-07
0.4	0.26253113	6.079e-06	2.155e-08
0.5	0.23269678	5.184e-06	1.730e-07
0.6	0.19682681	4.183e-06	2.978e-07
0.7	0.15524811	3.122e-06	3.679e-07
0.8	0.10832276	2.044e-06	3.569e-07
0.9	0.05643860	9.904e-07	2.408e-07
1.0	0.00000000	0.000e+00	0.000e+00

*Fig. 3. Results for Problem 3.*

3. Conclusions

This thesis has succeeded in developing, analysing and implementing a new method called Hybrid Cubic B-Spline Method (HCBSM) to numerically solve a general class of non-linear Singular Boundary Value Problems (SBVPs). Singular differential equations are numerically challenging because they are typically unbounded near the origin, and conventional discrete numerical methods often are not applicable, unstable or lose their accuracy at the origin. To resolve these limitations, the authors tackled the indeterminate form at the singular boundary $x=0$ systematically by directly using L'Hôpital's rule in

the spline formulation without the need to split the domain or to perform tedious coordinate transformations.

With the introduction of an adjustable blending parameter, the proposed hybrid approach showed much improved approximation power over the classical methods, while using the standard cubic B-splines. On similar mesh grids, computational experiments were performed on a number of different non-linear structures, such as algebraic, exponential and trigonometric, and it was found that HCBSM consistently outperforms the traditional Cubic B-Spline schemes by two to three orders of magnitude in terms of maximum absolute (L_∞) and root mean square (L_2) error norms. Moreover, an optimal fourth-order rate ($O(h^4)$) was obtained in all the singular indices considered in the experiments. The scheme when used together with Newton-Raphson quasilinearization, gave sparse tri-diagonal linear systems converging rapidly in a few iterations. In general, the Hybrid Cubic B-Spline Method is a highly accurate, stable and computational efficient method for solving the non-linear singular boundary value problems for solving applied mathematics, physics and engineering.

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