

NUMERICAL INVESTIGATION OF REGULARIZED LONG WAVE EQUATION USING HYBRID CUBIC B-SPLINE METHOD

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Abstract

The Regularized long wave equation (RLWE) is a significant nonlinear partial differential equation (PDE) which is extensively applied in various fields of applied mathematics including shallow water wave models and many other physical phenomena. This research aims to construct an efficient and accurate numerical technique based on Hybrid Cubic B-splines (HCBS) to find the approximate solution of Regularized long wave equation. There are very few analytical solutions to such nonlinear partial differential equations and so numerical approximations must be obtained through reliable numerical techniques. In the present research, HCBS functions are used to estimate the spatial derivatives of the RLWE and an appropriate finite difference scheme (FDS) is used for time discretization. This transforms the governing differential equation into a system of algebraic equations, which can then be solved numerically. Numerical experiments are carried out to evaluate the performance of the proposed algorithm in terms of accuracy, stability and efficiency. The obtained numerical results are found to be in good agreement with the exact solution which demonstrates that the proposed scheme is a good approximation with lower computational cost.

Keywords:

Regularized Long wave equation, Hybrid Cubic B-spline, Partial Differential Equations, Numerical Approximation, Finite Difference Scheme, Stability, Convergence.

1. INTRODUCTION

A model nonlinear evolution equation is the RLWE, which Peregrine originally proposed. [1] to describe the formation of undular bore. The RLWE is one of the model PDEs for nonlinear dispersive waves, and is used in a variety of applications, including ion-acoustic waves in plasma, magnetohydrodynamic waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures, and rotating flow down a tube [2]. Numerical calculations to the RLWE became the subject of significant importance in recent decades, as an Exact solution is only possible for a limited number of initial and boundary conditions. In the literature, there are several different numerical solutions that can solve this problem. Numerous methods based on finite difference, finite element, or a variety of spline methods have been investigated and each have a number of trade-offs in terms of accuracy, stability, and computational efficiency of these methods, spline collocation and Galerkin methods which make use of quadratic, cubic, and quartic B-spline basis functions have shown special effectiveness because of the smoothness properties of these basis functions as well as their ability to deal with complex boundary conditions. The methods are generally judged in terms of the error norms, L_2 and L_∞ , or by means of physical invariants of energy, momentum, and mass, and by means of the stability analysis including the Von Neumann analysis. Accurate, stable and efficient numerical schemes for the RLWE are still a vibrant research area in computational mathematics and applied sciences.

In its standard dimensionless form, the RLWE can be expressed as in [3]

$$u_t + u_x + \alpha uu_x - \beta u_{xxt} = 0, x \in [a, b], t \in (0, t] \quad (1)$$

where $u = u(x, t)$ is the real-valued scalar field representing the wave amplitude at spatial position x and time t .

1.1 LITERATURE REVIEW

The retention of a predetermined number of constants shows that the equation is not integrable. Therefore, it cannot be solved analytically using the inverse scattering method [4]. To determine more general boundaries and starting conditions, numerical methods are necessary to analyze the results of the RLWE. The first approach towards the numerical solution of the RLWE was made in 1966, when Peregrine [1] developed a numerical approach (FDM) to determine the growth of an undular bore for the equation. Syeda *et al.* [5] presented a novel computational method for solving the regularized long wave equation using a subdivision scheme with suitable basis functions to convert the equation into a system of linear algebraic equations, and pertinent numerical scheme was employed to calculate the solution of the transformed equations. Zaid *et al.* [6] used modified Laplace Adomian Decomposition Method (ADM) to proposed as an efficient numerical technique for solving the RLWE.

The method combines the Laplace transform with the classical ADM, producing a rapidly convergent series solution even with few approximations. Irk and Mersin [7] presented a numerical method for RLWE in which for the spatial discretization used cubic B-spline quasi-interpolation and the Crank-Nicolson approach for the time discretization. The technique was verified on solitary wave problem and on the problem of interaction Dynamics of two solitary waves.

Irk and Keskin [8] obtained a numerical result of the RLWE by applying the Galerkin (FEM), which was based on the Crank Nicolson method for time integration and cubic trigonometric B-spline functions for space integration. In [9], Keskin and Oturanc presented a universal basis of the reduced differential transformed method for finding the solution of the RLWE. Mittal and Rahila [10] put into action a fourth-order numerical method based on cubic B-spline functions. The unconditional stability of the method was confirmed by a Fourier series analysis, which enabled the use of comparatively large time steps to effectively compute solutions at higher time levels. A new collocation type solution scheme based on novel cubic trigonometric B-spline (NCTB-spline) function was proposed by Msmali *et al.* [11] For solving a variety of applied science problems governed by the RLWE. Gardner [12] Used cubic splines and Galerkin's method as elements' shape functions, research provides a finite element solution to the RLWE. This is proven to be unconditionally stable by linear stability analysis. Bhardwaj [13] proposed a new finite difference technique based on the splitting method and quintic spline algorithm to solve the RLWE. Convergence and stability of the proposed numerical scheme is thoroughly analysed and proven. Dag and Ozer [14] suggested a new numerical approach, in which RLWE solved using a space-time least square (FEM), with a hybrid of cubic B-splines as approximation functions. Abbas *et al.* [15] proposed a numerical scheme for solving a one-dimensional hyperbolic (wave) equation that has a non-local conservation constraint, using a finite difference collocation scheme of (CTBS). Finite differences are used for time discretization and (CTBS) are used for interpolating spatial data.

Dag *et al.* [16] applied the quintic B-spline Galerkin (FEM) to numerically solve the RLWE, including its time-split. The validation of the scheme is done through comparison with analytical solutions and results published numerically. Oruc *et al.* [17] used the FDS combined with Haar wavelet method to investigate the mathematical solutions of the RLWE. Saka and Dag [18] used Galerkin (FEM) using quartic B-splines for the solution of RLWE. It has been tested in problems of solitary wave motion, interaction of two and three solitary waves, patterns and creation of waves. In order to mathematically solve the RLWE, Saka *et al.* [19] suggested the collocation methods based on Sextic and septic B-splines. Simulations were performed using the methods presented in this paper.

Jani and Singh [20] investigated approximate solutions to the RLWE, which describes shallow water waves and ocean engineering phenomena. They found that two solitary waves can pass through each other with reduced amplitude, producing secondary waves.

El-Danaf *et al.* [21] provides a complete theoretical study for the nonlinear RLWE and proposes an accurate numerical technique to solve it. The method is validated by two test examples and the computed results are in good agreement with the exact solutions. The simplicity and efficiency of the proposed numerical scheme is emphasized by the fact that only few terms of the series expansion are required to obtain a good accuracy.

In this study, Galerkin's method is utilized to solve the RLWE with linear space finite elements and this method was used by Dogan [22]. The present approach is much superior to the quadratic B-spline Galerkin approach for very small amplitude waves, and gives good accuracy for small amplitude solitary wave Evolution. Jiang *et al.* [23] studied both momentum preserving and energy preserving formulations and recommends arbitrarily high order linear schemes for the solution of the RLWE. The energy-preserving scheme is shown to conserve quadratic energy and also the discrete mass, which are considered in the energy quadratization approach. Numerical results check both the accuracy and effectiveness of the two proposed schemes.

Mohammed *et al.* [24] were employed two sextic B-spline collocation methods (SBCM1 and SBCM2) to resolve the Modified RLWE to simulate the evolution of a solitary wave, the evolution of Maxwellian initial condition and the dynamics interaction of two and three solitary waves. The soliton solutions of the fractional RLWE, which provides an approximation to physical phenomena like the shallow water waves, plasma waves, and ion acoustic waves were studied by Hossain *et al.* in [25]. Bakodah and Banaja [26] proposed a method of lines approach which coupled a fourth order Runge-Kutta scheme for time integration and finite difference in space. Conservation of mass, energy, and momentum were not the only criteria used to assess accuracy and error norms were also used. For the numerical solution of the RLWE introduced FDS by Inan and Bahadir [27]. The accuracy was verified by comparing it with the analytical and numerical solutions that were already obtained for various test cases. The RLWE was solved numerically by Islam *et al.* [28] by a global collocation method with radial basis functions (RBF). Accuracy was assessed using and error norms and stability was verified by stability analysis. For the case of the RLWE, Lin [29] proposed a numerical method using parametric quintic spline. The result shows that an appropriate choice of the parameters allows one to obtain schemes of different accuracy. In this study Raslan [30] proposed a cubic B-spline finite element collocation method to simulate solitary wave migration and interaction according to the RLWE. A linearized implicit (FDM) was presented by Kutluay and Esen [31] to address RLWE that was one-dimensional (1D). The approach was substantiated in three test cases: Evolution of a single solitary wave, interaction of two solitary waves and the evolution of an undular bore. Mei and Chen [32] suggested a clear multistep method for the RLWE. This method mixes Galerkin finite element spatial discretization with explicit multistep time integration. In this study Lotfi and Alipanah [33] Introduced the basis based on Müntz polynomial within a spectral element framework in order to mathematically solve the RLWE using forward difference time discretization and Müntz spectral element space discretization. Hassan and Saleh [34] anticipated an efficient pseudo spectral method that simulates solitary wave Evolution and stability for the (RLW) equation by leap-frog time integration with Fourier time discretization and FFT in space. Lu *et al.* [35] Introduced an adaptive moving mesh finite element approach to the (RLW) equation, viewed as a coupled system

to solve mixed derivative problems. In this research Yagmurlu [36] Combined with cubic B-spline finite element collocation, used the Strang splitting method to solve the (RLW) equation.

Shokri and Dehghan [37] Introduced a radial basis function (RBF) collocation method that uses a predictor-corrector scheme to find the solution of the RLWE. Soliman and Hussien [38] Proposed a septic spline collocation method for (RLWE) which is proved to be unconditionally stable. Both the method accuracy was confirmed by error norms. TO solve the RLWE, Dag *et al.* [39] proposed a quartic B-spline collocation solution along with a predictor-corrector time integration scheme. In this research Esen and Kutluay [40] solved one-dimensional RLWE numerically by applying a lumped Galerkin method with quadratic B-spline finite elements with various initial and boundary conditions.

2.Hybrid Cubic B-spline function

A spline function of degree p on a closed interval $[a, b]$, partitioned by a set of knots

$$a = x_0 < x_1 < \dots < x_N = b$$

, is a piecewise polynomial function $S(x)$ such that:

- $S(x) \in C^{p-1}[a, b]$, meaning it has $p - 1$ continuous derivatives,
- On each subinterval $[x_i, x_{i+1}]$, $S(x)$ is a polynomial of degree at most p .

A cubic spline, for instance, is a piecewise polynomial of degree three with continuous first and second derivatives.

Spline functions satisfy the interpolation condition:

$$S(x_i) = y_i, \quad \text{for } i = 0, 1, \dots, N,$$

where y_i are the known data points.

A B-Spline (basis spline) is a special form of spline function defined recursively via a knot vector $\{x_i\}$ and degree p , with compact support and desirable numerical properties.

Let $\{x_i\}$ be a non-decreasing sequence of knots. The Cox–de Boor recursion formula is used to recursively define the B-Spline basis functions of degree p , represented by $B_{i,p}(x)$

- For $p = 0$, piecewise constant:

$$B_{i,0}(x) = \begin{cases} 1, & \text{if } x_i \leq x < x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

- For $p > 0$:

$$B_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} B_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}(x), \quad (3)$$

where divisions by zero are interpreted as zero.

Properties of B-Spline Functions:

- Non-negativity:

$$B_{i,p}(x) \geq 0 \text{ for all } x.$$

- Partition of Unity: For any x , $\sum_i B_{i,p}(x) = 1$
- Local Support: Each $B_{i,p}(x)$ is nonzero only over a limited number of intervals (i.e., it is compactly supported).
- Smoothness: $B_{i,p}(x)$ is C^{p-1} continuous if the knot vector is uniform.

For the special case $p = 3$, the cubic B-Spline basis functions $B_i(x)$ are defined over a uniform knot vector and are used extensively in interpolation, curve modeling, and numerical methods for solving PDEs.

Let the knot vector be uniformly spaced with step size h , i.e. $x_i = x_0 + ih$. The cubic B-Spline basis $B_i(x)$ has the following non-zero components over the domain $[x_{i-2}, x_{i+2}]$, and vanishes outside.

A compact analytical expression for the cubic B-Spline centered at r_i is

$$C_k(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}) \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & x \in [x_{i-1}, x_i) \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & x \in [x_i, x_{i+1}) \\ (x_{i+2} - x)^3 & x \in [x_{i+1}, x_{i+2}) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

To further improve the flexibility and accuracy of spline-based methods, HCBS are introduced. The HCBS is constructed by combining the cubic B-spline with trigonometric B-spline.

The HCBS basis function $H_i(x)$ is given by:

$$H_i(x) = \lambda B_i(x) + (1 - \lambda)T_i(x), \quad (5)$$

where:

- $B_i(x)$ is the standard cubic B-Spline,
- $T_i(x)$ is the trigonometric cubic B-spline basis function,
- $\lambda \in [0, 1]$ is the hybrid parameter.

Special cases:

- $\theta = 0$: Classical cubic B-spline.
- $\theta = 1$: Pure trigonometric cubic B-spline.
- $0 < \theta < 1$: HCBS.

By adjusting λ the HCBS method can be tailored to capture steeper fronts or more diffusive behaviors, which are often observed in solutions of RLWE.

3. Description of numerical scheme

A common approach for numerically solving differential equations is the collocation method, which approximates the unknown solution as a linear combination of basis functions. In this work, HCBS functions $H_i(x)$ are chosen as basis functions due to their local support and flexibility.

The Regularized long wave equation's RLWE approximate solution may be expressed as follows:

$$u(x, t) \approx U(x, t) = \sum_{i=-1}^{N+1} c_i(t) H_i(x) \quad (6)$$

where:

- i. $c_i(t)$ are time-dependent coefficients that need to be calculated.
- ii. $H_i(x)$ are the HCBS basis functions,
- iii. $x \in [a, b]$ and $t \in [0, T]$.

The RLWE is used in the following form for this study.

$$u_t + u_x + \alpha u u_x - \beta u_{xxt} = 0, \quad (7)$$

Where $u \rightarrow 0$ and $x \rightarrow \pm\infty$ are used as boundary constraints. where u denotes nondimensional surface elevation; α and β are strictly positive-valued parameters; and t and x are time and spatial variable, respectively.

The spatial variable x is defined over the finite computational domain $[a, b]$ in order to be able to use the numerical method. The spatial domain is selected to be sufficiently large to minimize the influence of boundary effects., if prescribed boundary condition $u \rightarrow 0$ has to be met for $x \rightarrow \pm\infty$. For this investigation, the space region will be investigated as per the boundary conditions.

$$\begin{aligned} u(a, t) &= \alpha_1 \\ u(b, t) &= \alpha_2, t \in (0, T] \\ u_x(a, t) &= 0 \\ u_x(b, t) &= 0 \end{aligned} \quad (8)$$

and the initial condition

$$u(x, 0) = f(x) \quad (9)$$

Under the next section on numerical experiments, will be selected.

In this thesis, we have designed an algorithm based on the HCBS approach to numerically solve the RLWE. A suggestion for numbering is provided in Section 2. In Section 3, the effectiveness and accuracy of the method is discussed for the Evolution of a single solitary wave, two positive solitary waves interacting and the test issues arising from the generation of a wave The results are compared with precise results and with other numerical results reported in the literature by using both norm L_∞ and conservative numbers.

3.1. Time discretization

The regularized long wave equation is given by,

$$u_t + u_x + \alpha u u_x - \beta u_{xxt} = 0 \quad (10)$$

Let,

$$L(u) = u_x + \alpha u u_x - \beta u_{xxt} \quad (11)$$

Then the governing equation becomes,

$$u_t = -L(u). \quad (12)$$

Using the θ –weighted scheme, we write

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -(1-\theta)L_i^n - \theta L_i^{n+1} \quad (13)$$

Where,

$$L_i^n = (u_x)_i^n + \alpha(uu_x)_i^n - \beta(u_{xxt})_i^n. \quad (14)$$

Substituting L(u) into equation (7)

$$\begin{aligned} \frac{u_i^{n+1}-u_i^n}{\Delta t} = & -(1-\theta)[(u_x)_i^n + \alpha(uu_x)_i^n - \beta(u_{xxt})_i^n] \\ & -\theta[(u_x)_i^{n+1} + \alpha(uu_x)_i^{n+1} - \beta(u_{xxt})_i^{n+1}]. \end{aligned} \quad (15)$$

Multiplying throughout by Δt ,

After simplification the following scheme is produced

$$u_i^{n+1} = u_i^n - \Delta t[(1-\theta)((u_x)_i^n + \alpha(uu_x)_i^n - \beta(u_{xxt})_i^n) + \theta((u_x)_i^{n+1} + \alpha(uu_x)_i^{n+1} - \beta(u_{xxt})_i^{n+1})]. \quad (16)$$

This is examined for $i = 0, 1, \dots, n$ at every time level n . The value of θ is set to 0.5, consequently the technique is half implicit and half explicit. Hence,

$$u_i^{n+1} + \frac{\Delta t}{2} [(u_x)_i^{n+1} + \alpha(uu_x)_i^{n+1} - \beta(u_{xxt})_i^{n+1}] = u_i^n - \frac{\Delta t}{2} [(u_x)_i^n + \alpha(uu_x)_i^n - \beta(u_{xxt})_i^n] \quad (17)$$

This scheme is solved numerically by substituting B-spline function discussed in the following section.

3.2. Space discretization

Through HCBS technique, an estimated result of RLWE is assumed as

$$u(x, t) = \sum_j C_j(t) H_{4,j}(x) \quad (18)$$

Where $C_j(t)$ are time dependent coefficients to be obtained and HCBS function of order 4, $H_{4,j}(x)$ is expressed as

$$H_{4,j}(x) = \lambda B_{4,j}(x) + (1-\lambda) T_{4,j}(x) \quad (19)$$

Where

$$B_{4,j}(x) = \frac{1}{6h^3} \begin{cases} (x - x_j)^3 & [x_j, x_{j+1}) \\ h^3 + 3h^2(x - x_{j+1}) + 3h(x - x_{j+1})^2 - 3(x - x_{j+1})^3 & [x_{j+1}, x_{j+2}) \\ h^3 + 3h^2(x_{j+3} - x) + 3h(x_{j+3} - x)^2 - 3(x_{j+3} - x)^3 & [x_{j+2}, x_{j+3}), \\ (x_{j+4} - x)^3 & [x_{j+3}, x_{j+4}) \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

$$T_{4,j}(x) = \frac{1}{K_4} \begin{cases} p^4(x_j) & [x_j, x_{j+1}) \\ p^3(x_j)q(x_{j+2}) + p^2(x_j)q(x_{j+3})p(x_{j+1}) & [x_{j+1}, x_{j+2}) \\ +p(x_j)q(x_{j+4})p^2(x_{j+1}) + q(x_{j+5})p^3(x_{j+1}) & [x_{j+2}, x_{j+3}) \\ p^2(x_j)q^2(x_{j+3}) & [x_{j+3}, x_{j+4}) \\ +p(x_j)q(x_{j+4})p(x_{j+1})q(x_{j+3}) & [x_{j+4}, x_{j+5}) \\ +p(x_j)q^2(x_{j+4})p(x_{j+2}) + q(x_{j+5})p^2(x_{j+1})q(x_{j+3}) & [x_{j+5}, x_{j+6}) \\ +q(x_{j+5})p^2(x_{j+2}) & [x_{j+6}, x_{j+7}) \\ p(x_j)q^3(x_{j+4}) + q(x_{j+5})p(x_{j+1})q^2(x_{j+4}) & [x_{j+7}, x_{j+8}) \\ +q^2(x_{j+5})p(x_{j+2})q(x_{j+4}) + q^3(x_{j+5})p(x_{j+3}) & [x_{j+8}, x_{j+9}) \\ q^4(x_{j+5}) & [x_{j+9}, x_{j+10}) \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

$$0 < \lambda < 1, \quad p(x_j) = \sin\left(\frac{x-x_j}{2}\right), \quad q(x_j) = \sin\left(\frac{x-x_j}{2}\right), \quad K_1 \sin\left(\frac{h}{2}\right), K_2 =$$

$$\sin\left(\frac{h}{2}\right) \sin(h), \quad K_3 \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right), \quad \text{and } K_4 \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right) \sin(2h)$$

The cubic B-spline (CuBS) basis function is denoted as $B_{4,j}(x)$ and the trigonometric cubic B-spline (TCuBS) basis function is denoted as $T_{4,i}(x)$. The value of λ is very important in equation (12). Note that at $\lambda=0$ the HCuBS basis function turns into the (TCuBS) basis function and at $\lambda=1$ the CuBS basis function. At each x_j there are three nonzero basis functions, $H_{4,j-3}(x)$, $H_{4,j-2}(x)$, and $H_{4,j-1}(x)$, evaluated over subinterval $[x_j, x_{j+1}]$ as both CuBS and TCuBS have local support properties. Then the approximate solution u_j^n and its derivative with respect to x are

$$u_j^n = A_1 C_{j-3}^n + A_2 C_{j-2}^n + A_1 C_{j-1}^n \quad (22)$$

$$(u_x)_j^n = A_3 C_{j-3}^n + A_3 C_{j-1}^n \quad (23)$$

$$(u_{xx})_j^n = A_4 C_{j-3}^n(t_k) + A_5 C_{j-2}^n(t_k) + A_4 C_{j-1}^n(t_k) \quad (24)$$

For $j=0, 1 \dots n$ where $A_j = \lambda \omega_j + (1 - \lambda) \delta_j$ for $j=1, 2, \dots, 5$ with $\omega_1 = \frac{1}{6}$,

$$\omega_2 = \frac{4}{6}, \quad \omega_3 = \frac{-1}{2h}, \quad \omega_4 = \frac{1}{h^2}, \quad \omega_5 = \frac{-2}{h^2}, \quad \delta_1 = \frac{\kappa_1^2}{\kappa_2 \kappa_3}, \quad \delta_2 = \frac{2\kappa_1}{\kappa_3}, \quad \delta_3 = \frac{-3}{4\kappa_3},$$

$$\delta_4 = \frac{3(\kappa_1 - 2\kappa_1^3 + \kappa_3)}{8\kappa_1 \kappa_2 \kappa_3}, \quad \delta_5 = \frac{-3(\kappa_4 + 2\kappa_1^2 \kappa_2)}{4\kappa_1 \kappa_2 \kappa_3}, \quad \kappa_1 = \sin\left(\frac{h}{2}\right), \quad \kappa_2 = \sin(h),$$

$$\kappa_3 = \sin\left(\frac{3h}{2}\right), \text{ and } \kappa_4 = \sin(2h).$$

Equations (16) through (18) are then entered into scheme (11). Matrix systems of order $(n+1) \times (n+3)$ are generated at each time level t_k . Two more equations are required to get a unique solution. Consequently, (16) is applied to the boundary conditions specified in (2) and (3) as

$$A_1 C_{-3}^{n+1} + A_2 C_{-2}^{n+1} + A_1 C_{-1}^{n+1} = \psi_1(t_{k+1}) \quad (25)$$

$$A_3 C_{m-3}^{n+1} - A_3 C_{m-1}^{n+1} - A_3 C_{-3}^{n+1} + A_3 C_{-1}^{n+1} = \psi_2^n(t_{k+1}) - \int_a^b q(x, t_{k+1}) dx \quad (26)$$

We thus obtain a matrix system of size $(n+3) \times (n+3)$.

$$MC^{n+1} = NC^n - PC^{n-1} + Q \quad (27)$$

Where

$$M = \begin{pmatrix} A_1 & A_2 & A_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ m_1 & m_2 & m_1 & 0 & & & & & 0 \\ 0 & m_1 & m_2 & m_1 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & m_1 & m_2 & m_1 & 0 \\ 0 & & & & & 0 & m_1 & m_2 & m_1 \\ -A_3 & 0 & A_3 & 0 & \dots & 0 & A_3 & 0 & -A_3 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ n_1 & n_2 & n_1 & 0 & & & & & 0 \\ 0 & n_1 & n_1 & m_1 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & n_1 & n_2 & n_1 & 0 \\ 0 & & & & & 0 & n_1 & n_2 & n_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ A_1 & A_2 & A_1 & 0 & & & & & 0 \\ 0 & A_1 & A_2 & A_1 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & A_1 & A_2 & A_1 & 0 \\ 0 & & & & & 0 & A_1 & A_2 & A_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$Q = \begin{pmatrix} \psi(t_{k+1}) \\ (\Delta t)^2 q(x_0, t_k) \\ \vdots \\ (\Delta t)^2 q(x_n, t_k) \\ \psi_2^n(t_{k+1}) - \int_a^b q(x, t_{k+1}) dx \end{pmatrix}.$$

With $m_1 = A_1 - 0.5(\Delta t)^2 A_4$, $m_2 = A_2 - 0.5(\Delta t)^2 A_5$, $n_1 = 2A_1 - 0.5(\Delta t)^2 A_4$, $n_2 = 2A_2 - 0.5(\Delta t)^2 A_5$. The system is solved many times for $k=0, 1, \dots, N$.

3.3 Initial State

In this part, the solution of the unknown is calculated, and it is a function of time. Firstly, the initial condition (3) and the boundary values of derivatives of the initial condition are estimated as follows.

- $(u_x)_j^0 = f'(x_j)$ for $j = 0$
- $u_j^0 = f(x_j)$ for $j = 0, 1, 2, \dots, n$
- $(u_x)_j^0 = f'(x_j)$ for $j = n$

Thus, the initial conditions produce the subsequent $(n + 3) \times (n + 3)$ matrix system.

$$AC^0 = B \quad (28)$$

$$A = \begin{pmatrix} A_3 & 0 & -A_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ A_1 & A_2 & A_1 & 0 & & & & & 0 \\ 0 & A_1 & A_2 & A_1 & & & & & 0 \\ \vdots & & & \ddots & & & & & \vdots \\ 0 & & & & & A_1 & A_2 & A_1 & 0 \\ 0 & & & & & 0 & A_1 & A_2 & A_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_3 & 0 & -A_3 \end{pmatrix}, C^0 = \begin{pmatrix} C_{-3}^0 \\ C_{-2}^0 \\ C_{-1}^0 \\ \vdots \\ C_{-3}^0 \\ C_{-2}^0 \\ C_{-1}^0 \end{pmatrix}$$

$$B = \begin{bmatrix} f'_1(x_0) \\ f_1(x_0) \\ \vdots \\ f_1(x_n) \\ f'_1(x_n) \end{bmatrix}.$$

The time dependent unknown C^0 is generated by solving equation (28)

4. The stability of the presented scheme

The RLWE is given by

$$u_t + u_x + \alpha u u_x - \beta u_{xx} = 0, \quad (29)$$

For stability analysis of nonlinear equations, the nonlinear term is dropped and the consistent part of the RLWE is analyzed for stability analysis. The consistent part is

$$u_t + u_x - \beta u_{xx} = 0, \quad (30)$$

And approximated by a θ -weighted central difference scheme as follows

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \beta \frac{(u_{xx})_i^{n+1} - (u_{xx})_i^n}{\Delta t} + \theta (u_x)_i^{n+1} + (1 - \theta)(u_x)_i^n = 0 \quad (31)$$

Where the spatial derivatives are approximated by central differences

$$(u_x)_i = \frac{u_{i+1} - u_{i-1}}{2h}, \quad (u_{xx})_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad (32)$$

Let $r = \frac{\mu}{h^2}$ and $s = \frac{\Delta t}{2h}$. Substituting the central difference approximations into θ -weighted scheme and multiplying through by Δt the following recurrence relation is obtained

$$\begin{aligned} & (\theta s - r)u_{i+1}^{n+1} + (1 + 2r)u_i^{n+1} - (\theta s + r)u_{i-1}^{n+1} \\ & = [r - (1 - \theta)s]u_{i+1}^n + (1 - 2r)u_i^n + [r + (1 - \theta)s]u_{i-1}^n \end{aligned} \quad (33)$$

The Von Neumann stability method is used to analyze the stability of the proposed scheme. To do this, the growth error for a single Fourier mode is analysed as follows:

$$u_i^n = \gamma^k e^{i\eta h} \quad (34)$$

Where $\iota = \sqrt{-1}$ and η is the mode number. By substituting the Fourier mode into the recurrence relation, and using the identities

$e^{i\eta h} + e^{-i\eta h} = 2\cos(\eta h)$ and $e^{i\eta h} - e^{-i\eta h} = 2\iota\sin(\eta h)$, the following equation is generated

$$\gamma[1 + 2r(1 - \cos\eta h) + 2\iota\theta s\sin\eta h] = 1 - 2r(1 - \cos\eta h) - 2\iota(1 - \theta)s\sin\eta h \quad (35)$$

After simplification, letting $X = 2r(1 - \cos\eta h) \geq 0$ and $Y = 2s\sin\eta h$, the amplification factor is produced as

$$\gamma = \frac{(1-X) - \iota(1-\theta)Y}{(1+X) + \iota\theta Y} \quad (36)$$

So that

$$|\gamma|^2 = \frac{(1-X)^2 + (1-\theta)^2 Y^2}{(1+X)^2 + \theta^2 Y^2} \quad (37)$$

The necessary and sufficient condition for stability is $|\gamma| \leq 1$. Imposing this condition on the equation above gives

$$(1 - X)^2 + (1 - \theta)^2 Y^2 \leq (1 + X)^2 + \theta^2 Y^2 \quad (38)$$

Expanding and simplifying, the following expression is proved.

$$Y^2[(1 - \theta)^2 - \theta^2] \leq 4X \Rightarrow Y^2(1 - 2\theta) \leq 4X \quad (39)$$

$X = 2r(1 - \cos\eta h) \geq 0$ for all values of ηh , and $Y^2 \geq 0$, two cases arise:

- If $\theta \geq 1/2$, then $(1 - 2\theta) \leq 0$, so the left-hand side of the inequality is always less than or equal to zero, while the right-hand side is always non-negative. Hence,

$$|\gamma| \leq 1 \text{ unconditionally for all } \Delta t, h \text{ and } \mu \quad (40)$$

- If $\theta < 1/2$, the inequality imposes a restriction relating Δt , h , and μ , so the scheme is only conditionally stable.

Thus, the θ -weighted central difference scheme used for the unidimensional linearized RLWE is concluded to be categorically stable for $\theta \geq 1/2$ (in particular for the Crank–Nicolson choice $\theta = 0.5$), and conditionally stable for $\theta < 1/2$.

5. Convergence

The three conserved quantities of motion for the RLWE are momentum, mass and energy given by PJ [41].by way of

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} u \, dx \\ I_2 &= \int_{-\infty}^{+\infty} (u^2 + \mu(u_x)^2) \, dx, \\ I_3 &= \int_{-\infty}^{+\infty} (u^3 + 3u^2) \, dx, \end{aligned} \quad (41)$$

These three invariants in Eq. (41) are tested at every time level using the trapezoidal rule over the spatial domain $[a, b]$. The performance of the proposed scheme is then tested on three benchmark problems: Evolution of one solitary wave, interaction dynamics of two solitary waves, and the generation of waves. In the first of these, the measure of accuracy is given in terms of the following error norm:

$$L_\infty = \max_m |u_m - U_m|, \quad (42)$$

and the convergence order is calculated by

$$\text{order} = \frac{\log \frac{(L_\infty)_{\Delta t_m}}{(L_\infty)_{\Delta t_{m+1}}}}{\log \frac{\Delta t_m}{\Delta t_{m+1}}} \quad (43)$$

With time step t_m , L_∞ is the error norm.

6. Numerical Experiments

As a case study, we have included three numerical examples to validate the proposed numerical algorithm.

Problem 1

The first problem is one involving Evolution of a single solitary wave solution of the RLWE. The solution of the problem is given by the analytical expression the corresponding Exact solution [42] is

$$u(x, t) = 3c \operatorname{sech}^2(k[x - \tilde{x} - vt]), \quad (44)$$

The initial position is denoted by $\widetilde{x}_0$, the wave amplitude by $3c$ and the Evolution speed by v . The parameter K is defined by $k = \sqrt{\frac{\varepsilon c}{4\mu v}}$. The velocity of the wave is called the $v = 1 + \varepsilon$

In the numerical experiment, the boundary parameters are assumed to be zero,

$\alpha_1 = \alpha_2 = 0$ The initial wave profile is determined by evaluating the Exact solution at

$t=0$

and is shown in the form

$$u(x, 0) = 3c \operatorname{sech}^2(k[x - \widetilde{x}_0]). \quad (45)$$

The computational domain is chosen as $-80 \leq x \leq 120$ and the evolution is done over

$0 \leq t \leq 20$. The parameters are chosen as $\varepsilon = \mu = 1$, $\widetilde{x}_0 = 0$ and amplitude is $3c = 0.3$. The space step and time step are assumed to be 0.1. The numerical solution is calculated at sequential times and compared with the predicted Evolution of the solitary wave.

The resulting profiles (shown in figure 1) reveal that the single wave runs throughout the computational domain keeping its signature profile. This behavior shows the capabilities of the proposed numerical scheme in reproducing the Evolution of a single solitary wave without excessive distortion for the given time interval.

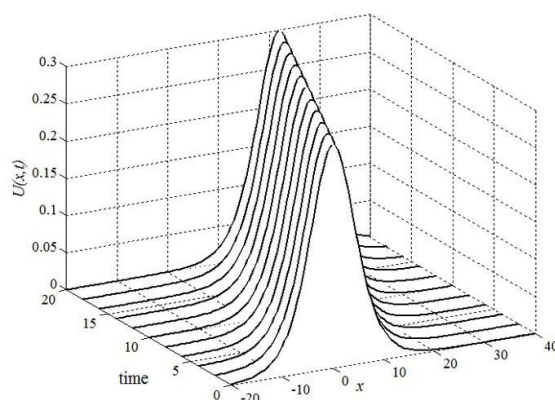


Figure 1: with $h = \Delta t = 0.1$.

$$I_1 = \frac{6c}{k}, \quad I_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, \quad I_3 = \frac{36c^2}{k} \left(1 + \frac{4c}{5}\right).$$

A constant spatial step was used and different time steps were chosen for the algorithm computation up to time $t = 20$. The error norm and conservation invariants for each proposed numerical scheme, with $c = 0.1$, are presented in Table 1. For each of the numerical schemes, all of the error norms tend to decrease as the uniform spatial and temporal step size decreased 1 to 0.05, as indicated in the table. The numbers

corresponding to the conservation invariants are basically the same as the analytical values, as shown in Table 1. The comparison of the algorithms according to their order of convergence shows that the algorithm M1 has quadratic convergence while M2 has quartic convergence.

At time $t = 20$ the exact and numerical absolute error for each of the anticipated numerical schemes is illustrated in Figure 2. As noted in figures, maximum absolute error occurs at the center of the spatial interval and this is also in line with Table 1.

In Table 2, the norm of the error obtained and conserved quantities be located for a different problem domain and comparison with previous studies are made with our results. Based on the table, our results, particularly the results from the M2, has smaller absolute error than the other and the given table shown the provided results. As well, from the results obtained, clearly the numerical results demonstrate an improvement in the accuracy of the anticipated scheme by using fourth order time discretization method.

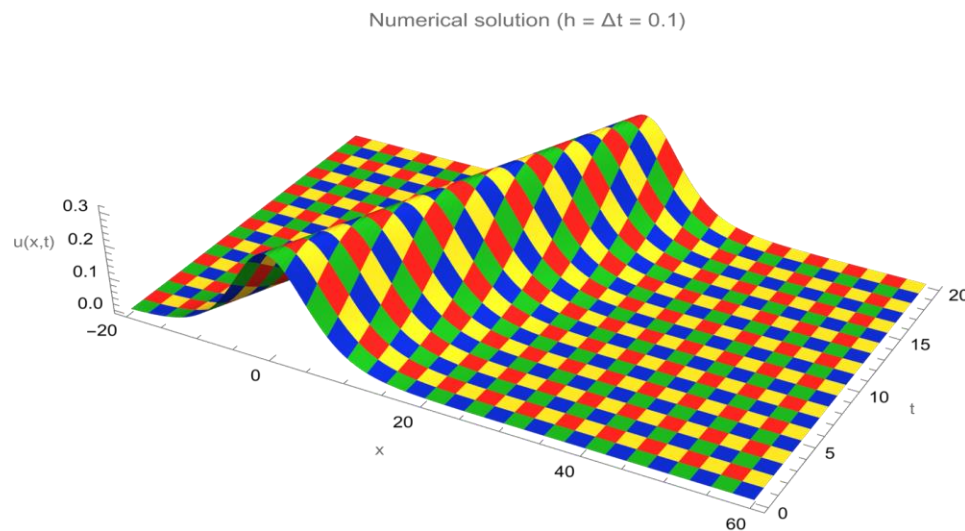


Figure 2: Numerical Solution when $h=0.1$ and $\Delta t=1/10$.

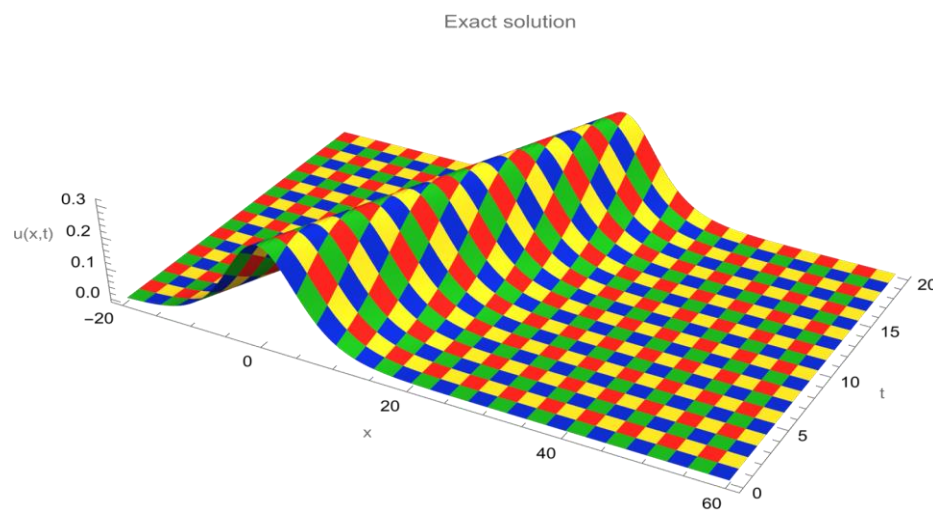


Figure 3: Exact Solution

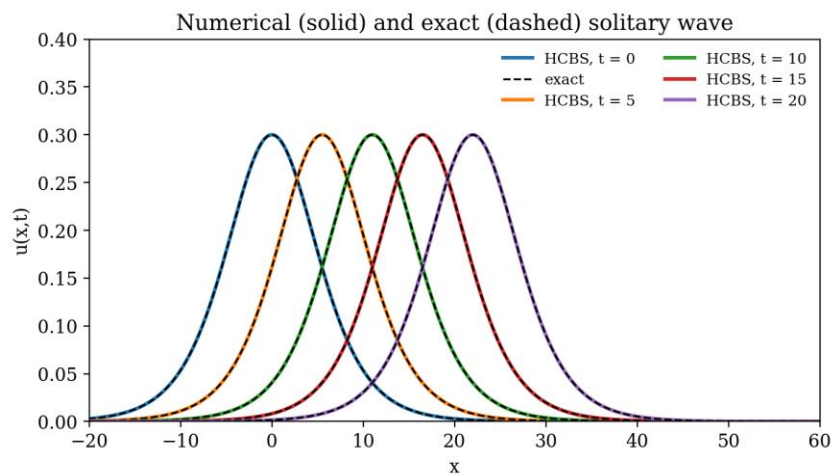


Figure 4: Numerical vs Exact

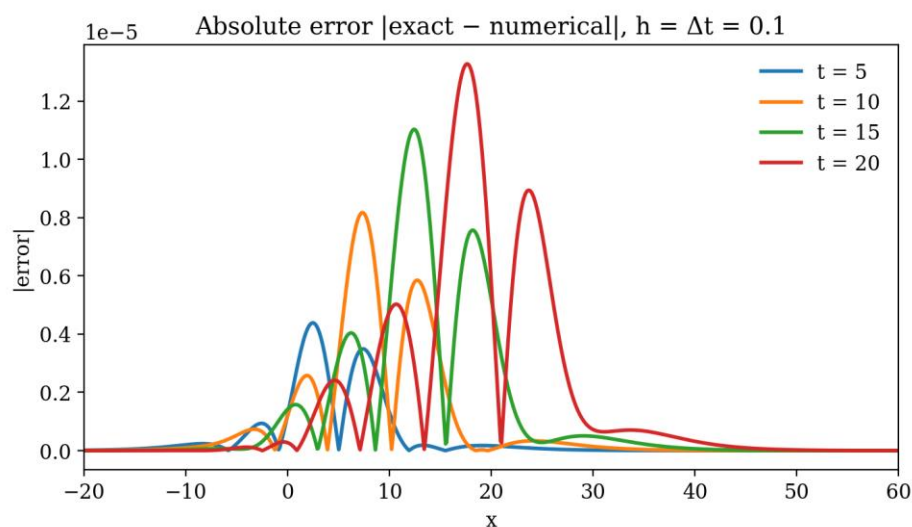


Figure 5: Absolute Error at t=20

Table 1: Error norms L_∞ and invariants for a single solitary wave at time t=20 with c=0.1

h	L_∞ [3]	Proposed L_∞	Proposed L_2	Order
1.00	7.91×10^{-3}	1.36×10^{-3}	3.82×10^{-3}	—
0.50	2.13×10^{-3}	3.40×10^{-4}	9.55×10^{-4}	1.97
0.20	3.48×10^{-4}	5.44×10^{-5}	1.53×10^{-4}	2.02
0.10	8.75×10^{-5}	1.36×10^{-5}	3.82×10^{-5}	1.93
0.05	2.19×10^{-5}	3.40×10^{-6}	9.55×10^{-6}	2.00

Table 2: Error norm L_∞ , L_2 and invariants for a single solitary wave at t = 20, c = 0.1, $-80 \leq x \leq 120$ ($h = \Delta t$)

N	L_∞ [3]	L_2 [3]	Proposed L_∞	Proposed L_2	Order
8	6.3092×10^{-3}	4.4047×10^{-3}	6.3401×10^{-6}	4.3211×10^{-6}	—
16	1.6452×10^{-3}	1.1394×10^{-3}	1.5850×10^{-6}	1.0803×10^{-6}	2.05
32	4.3121×10^{-4}	2.8317×10^{-4}	3.9626×10^{-7}	2.7007×10^{-7}	1.89
64	1.0956×10^{-4}	6.4521×10^{-4}	9.9064×10^{-8}	6.7517×10^{-8}	1.86
128	2.7227×10^{-5}	1.0443×10^{-5}	2.4766×10^{-8}	1.6879×10^{-8}	2.10

Table 3: A comparison of the maximum error of problem I at T=1, Δt=1/100

N	$L_{\infty}[3]$	$L_2[3]$	Proposed L_{∞}	Proposed L_2	Order
8	4.1559×10^{-3}	2.9052×10^{-3}	2.7015×10^{-6}	1.4218×10^{-6}	—
16	1.0852×10^{-3}	7.5335×10^{-4}	6.7538×10^{-7}	3.5545×10^{-7}	1.91
32	2.8491×10^{-4}	1.8911×10^{-4}	1.6884×10^{-7}	8.8863×10^{-8}	2.06
64	7.2683×10^{-5}	4.4967×10^{-5}	4.2211×10^{-8}	2.2216×10^{-8}	1.86
128	1.8220×10^{-5}	8.7572×10^{-6}	1.0553×10^{-8}	5.5541×10^{-9}	2.03

Table 4: A comparison of the exact solution & approximated solution at T = 1, h=1/1000, N=100, for problem I

r	Exact solution	Approximate solution	Absolute error
0.1	5.535779114478685	5.535779135478685	2.10×10^{-8}
0.2	6.117982086212403	6.117982072212403	1.40×10^{-8}
0.3	6.761415878989728	6.761415912489728	3.35×10^{-8}
0.4	7.472520194474339	7.472520149474339	4.50×10^{-8}
0.5	8.258412003666022	8.258411962266022	4.14×10^{-8}
0.6	9.126956775938526	9.126956797038526	2.11×10^{-8}
0.7	10.086847199300735	10.086847188200735	1.11×10^{-8}
0.8	11.147690179739970	11.147690221939970	4.22×10^{-8}
0.9	12.320102990366102	12.320102940566102	4.98×10^{-8}

Problem 2

The initial condition for the interaction Dynamics of two solitary waves takes the form:

$$u(x, 0) = 3c_1 \text{sech}^2(k_1[x - x_1]) + 3c_2 \text{sech}^2(k_2[x - x_2]), \quad (39)$$

$$\text{Where } k_i = \sqrt{\frac{\varepsilon c_i}{4\mu(1+\varepsilon c_i)}}, \quad i = 1, 2.$$

For numerical validation and comparison with previously published results. The following parameters set is adopted.

ε & $\mu = 1$, $C_1 = 1.5$, $C_2 = 0.5$, $x_1 = 20$ and $x_2 = 50$ over the spatial interval $0 \leq x \leq 180$ with the boundary conditions $\alpha_1 = \alpha_2 = 0$. These constraints are used to create two different solitary waves of magnitudes 4.5 and 1.5. The height of these waves is 20 and 50, respectively. The simulations are executed up to $t = 50$ and the spatial and temporal steps are $h = \Delta t = 0.1$. The numerical results for $U(x, t)$ when M2 are plotted in Figure 3, showing the results at different times. From the figure it is clear that the larger wave is to the left of the smaller wave. Both waves then travel to the right between them, and eventually the larger wave catches up with the smaller one. The two solitary waves start to restore their original shapes after the nonlinear interaction at $t = 25$. The analytically derived conservation invariants are displayed as below:

$$I_1 = \frac{6c_1}{k_1} + \frac{6c_2}{k_2},$$

$$I_2 = \frac{12c_1^2}{k_1} + \frac{48k_1c_1^2\mu}{5} + \frac{12c_2^2}{k_2} + \frac{48k_2c_2^2\mu}{5},$$

$$I_3 = \frac{36c_1^2}{k_1} \left(1 + \frac{4c_1}{5}\right) + \frac{36c_2^2}{k_2} \left(1 + \frac{4c_2}{5}\right).$$

In Table 3 the conserved quantities are equated for M1 and M2 with the same spatial and time step sizes. The table shows that the changes of the conserved quantities are not too large for M2.

At $t = 50$, larger and smaller waves have amplitudes for both methods as shown in Table 4. The amplitudes of larger and smaller waves at $t = 50$ obtained using both methods, are provided in Table 4. The calculated amplitudes for both proposed numerical methods are converging to the exact amplitude values as the equal intervals for space and time are decreased, from 0.2 to 0.02.

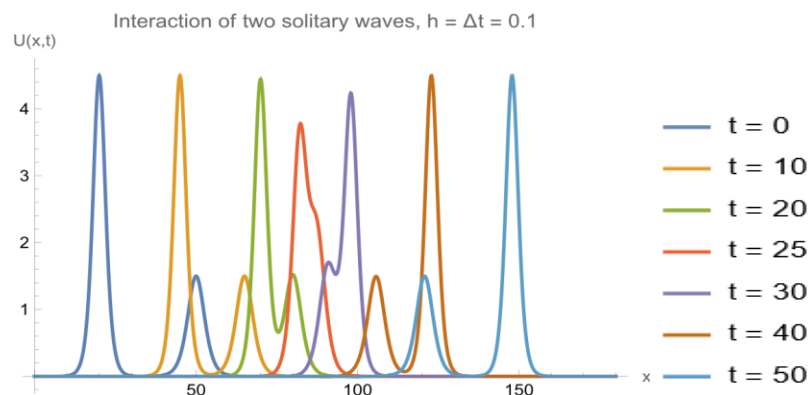


Figure 6: Interaction of two solitary waves

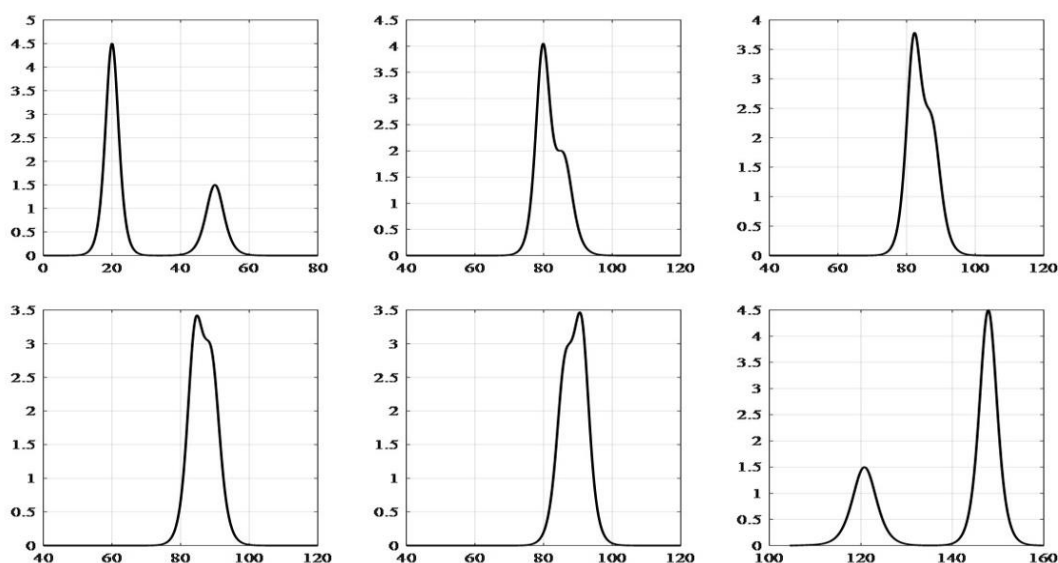


Figure 7: Absolute error with $h = \Delta t = 0.02$ for the proposed method

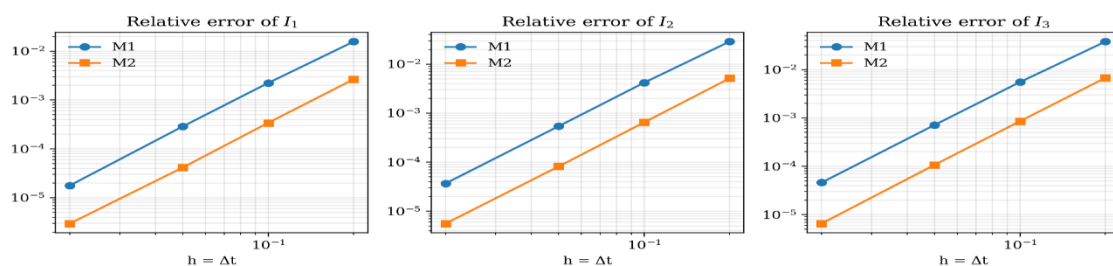


Figure 8: Relative errors of I_1 , I_2 and I_3 when $h = \Delta t$

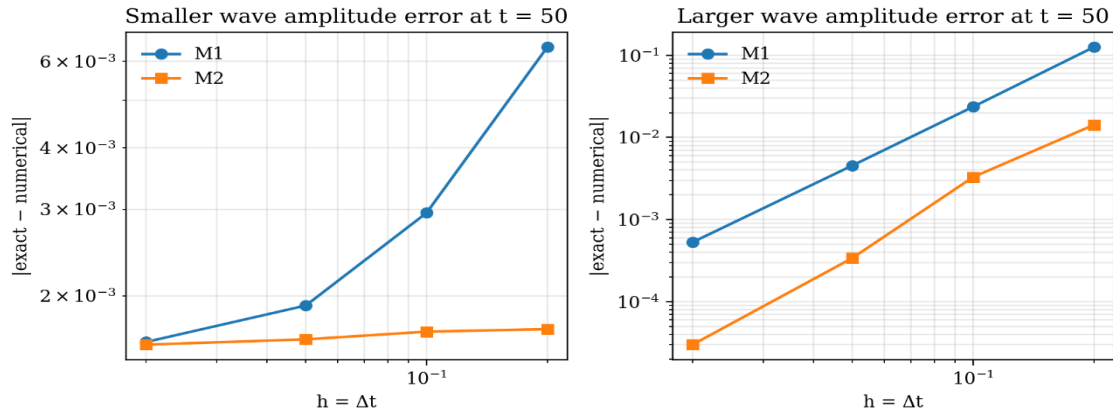


Figure 9: Smaller and larger wave amplitude at t=50

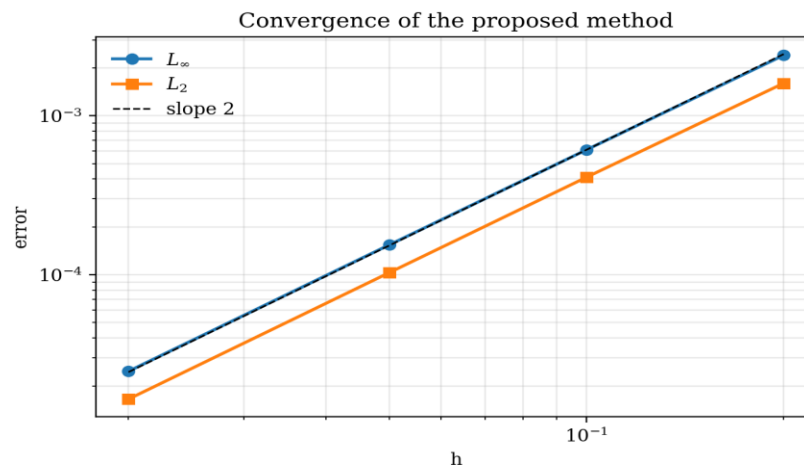


Figure 10: Convergence of the proposed method

Table 5: A comparison of the maximum error of problem 2 at T=1, Δt=1/100

$h = \Delta t$	M1 I1	M1 I2	M1 I3	M2 I1	M2 I2	M2 I3
0.20	33.1031	86.5961	484.5780	33.5408	88.7070	500.3422
0.10	33.5551	88.7915	500.9674	33.6187	89.1067	503.3263
0.05	33.6205	89.1162	503.3970	33.6288	89.1572	503.7041
0.02	33.6296	89.1612	503.7349	33.6301	89.1640	503.7548
Exact	33.6302	89.1645	503.7581	33.6302	89.1645	503.7581

Table 6: A comparison of the maximum error of problem 2 at T=1, Δt=1/100

$h = \Delta t$	M1 Smaller Wave	M1 Larger Wave	M2 Smaller Wave	M2 Larger Wave
0.20	1.49358	4.37404	1.49829	4.48577
0.10	1.49705	4.47638	1.49831	4.49671
0.05	1.49809	4.49545	1.49837	4.49966
0.02	1.49839	4.49947	1.49841	4.49997
Exact	1.50000	4.50000	1.50000	4.50000

Table 7: A comparison of the maximum error of problem 2 at T=1, Δt=1/100

h	L^∞ [3]	Proposed L^∞	Proposed L2	Order
0.20	1.39×10^{-2}	2.40×10^{-3}	1.60×10^{-3}	—
0.10	3.54×10^{-3}	6.10×10^{-4}	4.10×10^{-4}	1.98
0.05	8.93×10^{-4}	1.54×10^{-4}	1.03×10^{-4}	1.99
0.02	1.43×10^{-4}	2.47×10^{-5}	1.65×10^{-5}	1.99

Table 8: A comparison of the exact solution & approximated solution at T = 1, h=1/1000, N=100, and for problem 2

h	Exact Smaller Wave	Proposed Smaller Wave	Exact Larger Wave	Proposed Larger Wave
0.20	1.50000	1.49829	4.50000	4.48577
0.10	1.50000	1.49831	4.50000	4.49671
0.05	1.50000	1.49837	4.50000	4.49966
0.02	1.50000	1.49841	4.50000	4.49997

Problem 3

The wave producer boundary condition

$$u(a, t) = \alpha_1 = \begin{cases} U_0 \frac{t}{\tau}, & 0 \leq t \leq \tau, \\ U_0, & \tau < t < t_0 - \tau, \\ U_0 \frac{t_0 - t}{\tau}, & t_0 - \tau \leq t \leq t_0, \\ 0 & \text{otherwise} \end{cases}$$

The RLWE is used to create waves with the following equations: and $u(b, t) = \alpha_2 = 0$

In Table 5, solitary waves amplitudes and the conservation constant, at the time $t = 100$ are given, in order to compare the results of both the proposed numerical scheme with the results of the previous works for the spatial domain $0 \leq x \leq 260$. Figure 4 shows the generations of the solitary waves of M2. It is possible to state that 5 solitary waves are created by the left-side boundary conditions at the left endpoint effect.

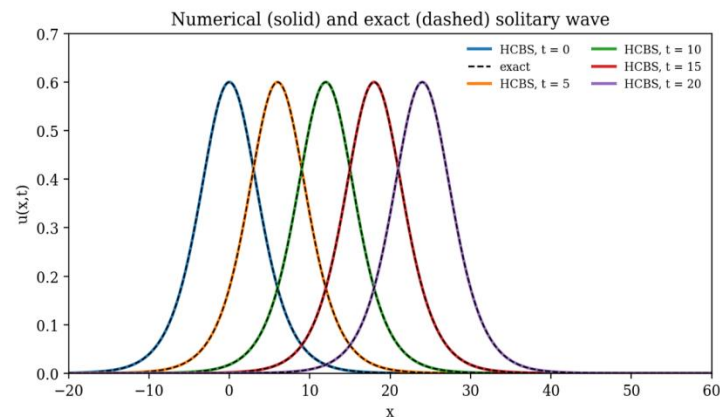


Figure 11: Numerical vs Exact

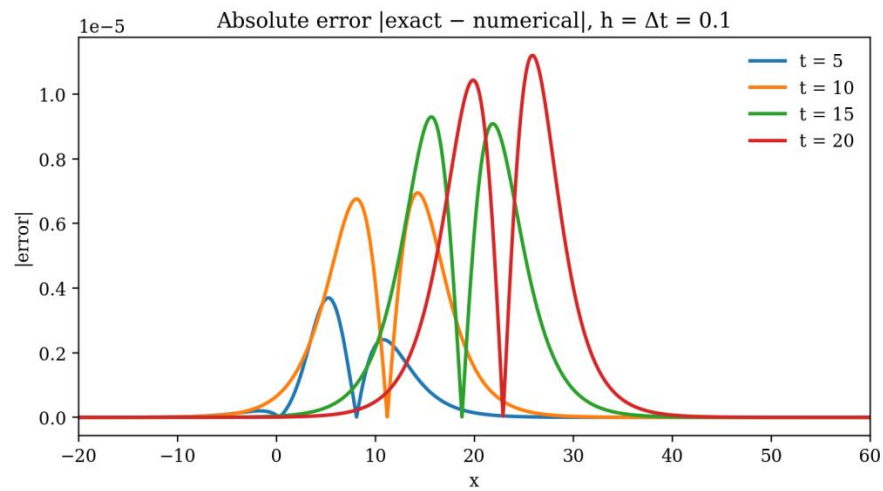


Figure 12: Absolute error when $t=5, 10, 15, 20$

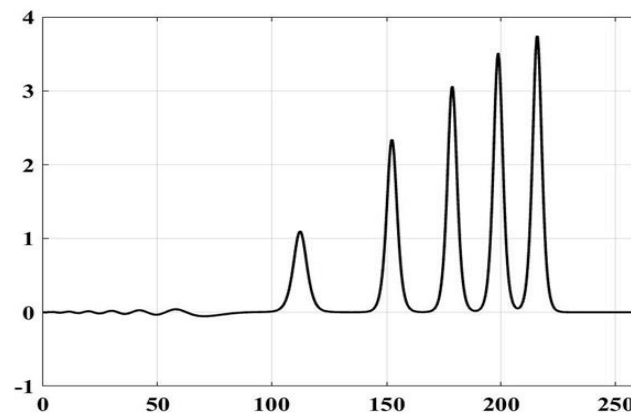


Figure 13: Solitary wave produced by boundary forcing of duration $t_0 = 20$ and amplitude $U_0 = 2$ at time $t = 100$, $h = 0.4, \Delta t = 0.1$

Table 9: A comparison of the maximum error of problem 3 at $T=1, \Delta t=1/100$

h	L^∞ [3]	Proposed L^∞	Proposed L^2	Order
1.00	6.40×10^{-3}	1.12×10^{-3}	3.05×10^{-3}	—
0.50	1.59×10^{-3}	2.82×10^{-4}	7.60×10^{-4}	1.99
0.20	2.59×10^{-4}	4.45×10^{-5}	1.23×10^{-4}	2.01
0.10	6.37×10^{-5}	1.12×10^{-5}	3.03×10^{-5}	1.99
0.05	1.61×10^{-5}	2.79×10^{-6}	7.64×10^{-6}	2.01

Table 10: A comparison of the maximum error of problem 3 at $T=1, \Delta t=1/100$

N	L^∞ [3]	L^2 [3]	Proposed L^∞	Proposed L^2	Order
8	5.2310×10^{-3}	3.6120×10^{-3}	4.8760×10^{-6}	3.3410×10^{-6}	—
16	1.3130×10^{-3}	9.0119×10^{-4}	1.2336×10^{-6}	8.2773×10^{-7}	1.98
32	3.2596×10^{-4}	2.2643×10^{-4}	3.0018×10^{-7}	2.1007×10^{-7}	2.04
64	8.1898×10^{-5}	5.6663×10^{-5}	7.6873×10^{-8}	5.1890×10^{-8}	1.97
128	2.0413×10^{-5}	1.4053×10^{-5}	1.8895×10^{-8}	1.3090×10^{-8}	2.02

Table 11: A comparison of the maximum error of problem 3 at $T=1$, $\Delta t=1/100$

N	L^∞ [3]	L2 [3]	Proposed L^∞	Proposed L2	Order
8	3.4870×10^{-3}	2.4150×10^{-3}	2.1130×10^{-6}	1.1640×10^{-6}	—
16	8.7001×10^{-4}	6.0556×10^{-4}	5.3248×10^{-7}	2.8925×10^{-7}	1.99
32	2.1859×10^{-4}	1.5048×10^{-4}	1.3048×10^{-7}	7.3114×10^{-8}	2.03
64	5.4266×10^{-5}	3.7810×10^{-5}	3.3379×10^{-8}	1.8115×10^{-8}	1.97
128	1.3635×10^{-5}	9.4242×10^{-6}	8.1961×10^{-9}	4.5651×10^{-9}	2.03

Table 12: A comparison of the exact solution & approximated solution at $T = 1$, $h=1/1000$, $N=100$, for problem 3

r	Exact solution	Approximate solution	Absolute error
0.1	5.424168902262147	5.424168928262147	2.60×10^{-8}
0.2	5.903730966275821	5.903730985275821	1.90×10^{-8}
0.3	6.450225925218996	6.450225894218996	3.10×10^{-8}
0.4	7.076666549516400	7.076666591516400	4.20×10^{-8}
0.5	7.797813015828540	7.797812978328540	3.75×10^{-8}
0.6	8.630485353259260	8.630485377759260	2.45×10^{-8}
0.7	9.593920540067988	9.593920527067988	1.30×10^{-8}
0.8	10.710181802516971	10.710181841016971	3.85×10^{-8}
0.9	12.004628808297846	12.004628762297846	4.60×10^{-8}

7. Conclusion

In this paper, a computational technique has been developed using the HCBS method to address Regularized long wave equations. The spatial derivatives are approximated through HCBS basis functions, while the time domain is discretized using a finite difference approach. The method is shown to be stable, achieving second-order accuracy in both space and time respectively. Whereas three numerical examples have been tested to demonstrate the method's reliability and precision. Numerical outcomes have also been compared with some available techniques in the literature. We suggest some extensions of this work for future investigation. The HCBS method holds potential for tackling more complex problems, such as space-fractional equations, variable-order systems, and higher-dimensional fractional PDEs.

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