

NUMERICAL SOLUTIONS OF EQUAL WIDTH EQUATION USING HYBRID CUBIC B-SPLINE METHOD

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Abstract

This study presents a strong and very accurate numerical solution of the non-linear Equal Width (EW) equation. The numerical method proposed here is based on a finite difference method for the time discretization and a Hybrid Cubic B-spline (HCBS) spatial approximation method. The Rubin and Graves quasilinearization method is used to handle the non-linear term that is part of the EW equation. Proposed finite difference scheme is rigorously proven to be unconditionally stable by applying Von Neumann (Fourier) stability analysis method. Moreover, a formal convergence analysis is presented that shows that the second-order accuracy is reached both in time and space $O(\Delta t^2 + h^2)$. For the validation of the theoretical framework three classic numerical examples are studied: propagation of a single solitary wave, interaction of two solitary waves and the evolution of wave trains from a Maxwellian initial condition. Three conserved motion invariants (I_1, I_2, I_3) are used to evaluate the numerical performance along with the error norms (L_2) and (L_∞). Numerical results show the superiority of the Hybrid Cubic B-spline scheme in terms of spatial resolution and superior conservation properties, over the standard B-spline schemes reported in the literature.

Keywords:

Equal Width equation; Hybrid Cubic B-spline; Finite difference scheme; Solitary waves; Von Neumann stability; Conserved invariants.

1. INTRODUCTION

Non-linear partial differential equations (PDEs) are the basic mathematical models in the physical sciences and engineering, especially for fluid mechanics, plasma physics, non-linear optics and wave dynamics [1]–[3]. The Equal Width (EW) equation, proposed by Morrison et al, is one of non-linear dispersive wave equations. One of the non-linear dispersive wave equations is the Equal Width (EW) equation proposed by Morrison et al. [4] in 1984. It is an important parameter to describe one-dimensional non-linear wave propagation in dispersing media. The EW equation is an alternative model to the classical Korteweg-de Vries (KdV) equation and the Regularized Long Wave (RLW) equation. [5]–[7].

The non-linear dimensionless governing equation is:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \mu \frac{\partial^3 u}{\partial x^2 \partial t} = 0, \quad a \leq x \leq b, t > 0 \quad (1)$$

subject to boundary conditions:

$$u(a, t) = 0, \quad u(b, t) = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=a} = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=b} = 0 \quad (2)$$

and initial condition:

$$u(x, 0) = f(x), \quad a \leq x \leq b \quad (3)$$

In the above, $u(x, t)$ is the amplitude of the wave, α is a positive non-linear parameter, and μ is the dispersion coefficient. The name "Equal Width" is taken from the physical property of the solitary wave solution to Eq. (1) that the wave width is constant for any wave amplitude. [8], [9].

There are many situations where analytic solutions to non-linear wave equations are not easily found, particularly with complicated initial or boundary conditions. This has led to a significant research interest in numerical methods. Numerous numerical methods have been suggested for the EW equation such as differential quadrature methods. [10], Galerkin finite element methods [11], The standard cubic B-splines were used for collocation methods. [12], quartic B-splines [13], and trigonometric B-splines [14].

While cubic B-splines are very simple to compute and are C^2 continuous, they can introduce numerical damping and phase errors in the region of steep gradients or peak amplitudes. [15]. To address these shortcomings, the Hybrid Cubic B-spline (HCBS) formulation has been developed, which is a combination of standard polynomial cubic B-splines and trigonometric or blend functions by using a continuous blending parameter $\gamma \in [0, 1]$. To overcome these shortcomings, a new spatial discretization method, Hybrid Cubic B-spline (HCBS) formulation that is the combination of the standard polynomial cubic B-spline and the trigonometric or blend functions, has been developed by using a continuous blending parameter. [16], [17].

The goal of this paper is to create, analyze and implement a high precision numerical scheme that includes a Crank-Nicolson temporal finite difference discretization and Hybrid Cubic B-spline spatial

collocation discretization for the solution of the non-linear EW equation. The non-linear advection terms are linearized to avoid compromising temporal accuracy using Rubin-Graves quasilinearization. In Section 2, the mathematical properties and conserved quantities of the EW equation are discussed. The Hybrid Cubic B-spline basis functions are presented in Section 3. The finite difference temporal discretization and system construction are given in Section 4. Section 5 offers Stability proofs, and Section 6 Convergence proofs. Three detailed numerical examples with tables, figures, and discussions are given in Section 7. Section 8 is the last section of the paper.

2. Mathematical Properties & Conserved Quantities

The EW equation (1) has three basic conservation laws: mass, momentum (energy), and non-linear wave energy. [18]:

1. Mass Invariant I_1 :

$$I_1 = \int_a^b u(x, t) dx \quad (4)$$

2. Momentum / Energy Invariant (I_2):

$$I_2 = \int_a^b \left[u(x, t)^2 + \mu \left(\frac{\partial u}{\partial x} \right)^2 \right] dx \quad (5)$$

3. Third Invariant (I_3):

$$I_3 = \int_a^b u(x, t)^3 dx \quad (6)$$

The analytical single solitary wave solution of Eq. (1) is given by:

$$u(x, t) = 3c \operatorname{sech}^2(k(x - vt - x_0)) \quad (7)$$

In which $v = c$ is the wave speed, x_0 is the initial wave center and the wave parameter k is given by

$$k = \sqrt{\frac{3c}{4\mu}} \quad (8)$$

3. Hybrid Cubic B-Spline (HCBS) Spatial Discretization

Suppose that the interval $[a, b]$ is divided into n equal parts, using n points $a = x_0, x_1, \dots, x_N, x_{N+1} = b$. $x_j = a + jh$, L , spatial step size and

$$h = (b - a) / N$$

The Hybrid Cubic B-spline basis functions are introduced $H_i(x)$ ($i = -1, 0, 1, \dots, N+1$). A linear combination of polynomial cubic B-splines is defined over the domain. $B_i(x)$. The cubic trigonometric B-splines $T_i(x)$ are introduced with the help of a weight parameter. $\gamma \in [0, 1]$:

$$H_i(x) = \gamma B_i(x) + (1 - \gamma) T_i(x) \quad (9)$$

At the nodes, the values of $H_i(x)$ and its first and second derivatives are equal. The nodal values of $x = x_j$ are given by:

$$H_i(x_j) = \begin{cases} a_1, & |j - i| = 1 \\ a_2, & j = i \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

$$\left. \frac{dH_i(x)}{dx} \right|_{x=x_j} = \begin{cases} -b_1, & j - i = 1 \\ b_1, & j - i = -1 \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

$$\left. \frac{d^2 H_i(x)}{dx^2} \right|_{x=x_j} = \begin{cases} c_1, & |j - i| = 1 \\ c_2, & j = i \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

The nodal parameters a_1, a_2, b_1, c_1, c_2 are dependent on h and blending factor γ :

$$a_1 = \frac{1 - \gamma}{24} + \frac{\gamma}{6}, \quad a_2 = \frac{11(1 - \gamma)}{12} + \frac{2\gamma}{3} \quad (13)$$

$$b_1 = \frac{1}{2h}, \quad c_1 = \frac{1 - \gamma}{h^2}, \quad c_2 = \frac{-2 + \gamma}{h^2} \quad (14)$$

The approximation to the numerical solution $U(x, t)$ and the spatial derivatives of $U(x, t)$ are approximated using the HCBS basis functions as:

$$U(x, t) = \sum_{i=-1}^{N+1} C_i(t) H_i(x) \quad (15)$$

Consequently, the nodal values U_j^n , $(U_x)_j^n$, and $(U_{xx})_j^n$ at $x = x_j$ and $t = t_n$ are expressed as:

$$U_j^n = a_1 C_{j-1}^n + a_2 C_j^n + a_1 C_{j+1}^n \quad (16)$$

$$(U_x)_j^n = -b_1 C_{j-1}^n + b_1 C_{j+1}^n \quad (17)$$

$$(U_{xx})_j^n = c_1 C_{j-1}^n + c_2 C_j^n + c_1 C_{j+1}^n \quad (18)$$

4. Temporal Discretization & Finite Difference Scheme

Using the Crank-Nicolson finite difference scheme in the temporal direction, the time step size is Δt ($t_n = n\Delta t$), Eq. (1) becomes:

$$\frac{U^{n+1}-U^n}{\Delta t} + \frac{\partial}{\partial x} \left[(UU_x)^{n+1} + (UU_x)^n \right] - \mu \frac{U_{xx}^{n+1} - U_{xx}^n}{\Delta t} = 0 \quad (19)$$

To address non-linearity without iteration costs this is accomplished by the Rubin and Graves quasilinearization technique.

$$(UU_x)^{n+1} \approx U^n U_x^{n+1} + U^{n+1} U_x^n - U^n U_x^n \quad (20)$$

After substituting Eq. (20) into Eq. (19), the linearized fully discrete scheme is obtained as follows:

$$\left(1 - \mu \frac{\partial^2}{\partial x^2} + \frac{\partial \Delta t}{2} U^n \frac{\partial}{\partial x} + \frac{\partial \Delta t}{2} U_x^n \right) U^{n+1} = \left(1 - \mu \frac{\partial^2}{\partial x^2} - \frac{\partial \Delta t}{2} U^n \frac{\partial}{\partial x} \right) U^n \quad (21)$$

Using the HCBS representations from Eqs. Substituting (16)–(18) into (21) gives a tri-diagonal linear matrix system of unknown control parameters. $\mathbf{C}^{n+1} = [C_{-1}^{n+1}, C_0^{n+1}, \dots, C_{N+1}^{n+1}]^T$:

$$\alpha_j^n C_{j-1}^{n+1} + \beta_j^n C_j^{n+1} + \gamma_j^n C_{j+1}^{n+1} = \hat{\alpha}_j^n C_{j-1}^n + \hat{\beta}_j^n C_j^n + \hat{\gamma}_j^n C_{j+1}^n \quad (22)$$

where the coefficients are given by:

$$\alpha_j^n = a_1 - \mu c_1 - \frac{\partial \Delta t}{2} b_1 U_j^n + \frac{\partial \Delta t}{2} a_1 (U_x)_j^n \quad (23)$$

$$\beta_j^n = a_2 - \mu c_2 + \frac{\partial \Delta t}{2} a_2 (U_x)_j^n \quad (24)$$

$$\gamma_j^n = a_1 - \mu c_1 + \frac{\partial \Delta t}{2} b_1 U_j^n + \frac{\partial \Delta t}{2} a_1 (U_x)_j^n \quad (25)$$

$$\hat{\alpha}_j^n = a_1 - \mu c_1 + \frac{\partial \Delta t}{2} b_1 U_j^n \quad (26)$$

$$\hat{\beta}_j^n = a_2 - \mu c_2 \quad (27)$$

$$O(\Delta t^2 + h^2) \quad \hat{\gamma}_j^n = a_1 - \mu c_1 - \frac{\partial \Delta t}{2} b_1 U_j^n \quad (28)$$

At two additional points (points of boundary conditions) there are two more equations based on the condition in Eq. (2). $j = -1$ and $j = N+1$, forming a closed $(N+3) \times (N+3)$ matrix system that is efficiently solved using Thomas algorithm.

5. Stability Analysis

The Von Neumann (Fourier) stability method is used to consider the stability of the proposed Temporal finite difference scheme and Hybrid Cubic B-spline spatial discretization.

Let U^n The advection coefficient in the non-linear PDE be locally linearized by assuming local constant velocity $Z = \max_j |U_j^n|$. The error mode at time step n of control parameter C_j^n is assumed to be:

$$C_j^n = \xi^n e^{ikjh}$$

where $i = \sqrt{-1}$, k is the spatial wave number, and ξ denotes the complex amplification factor. Inserting Eq. (29) in the linearized system (22) results in:

$$\left[(a_2 + 2a_1 \cos \theta) - \mu(c_2 + 2c_1 \cos \theta) + i\Delta t Z b_1 \sin \theta \right] \xi^{n+1} = \left[(a_2 + 2a_1 \cos \theta) - \mu(c_2 + 2c_1 \cos \theta) - i\Delta t Z b_1 \sin \theta \right] \xi^n \quad (31)$$

where $\theta = kh$. Defining:

$$\begin{aligned} X &= (a_2 + 2a_1 \cos \theta) - \mu(c_2 + 2c_1 \cos \theta) \\ Y &= \Delta t Z b_1 \sin \theta \end{aligned} \quad (32)$$

The amplification factor ratio $\xi = \frac{\xi^{n+1}}{\xi^n}$ is expressed as:

$$\xi = \frac{X - iY}{X + iY} \quad (33)$$

Taking the modulus of ξ :

$$|\xi|^2 = \frac{X^2 + Y^2}{X^2 + Y^2} = 1 \Rightarrow |\xi| = 1 \quad (34)$$

Since $|\xi| = 1$ for all spatial wave numbers, the numerical scheme is unconditionally stable

6. Convergence Analysis

Theorem 1: Let $u(x, t) \in C^4[a, b]$ be the exact solution of the Equal Width equation (1) and let $U(x, t)$ be the numerical solution computed by the Hybrid Cubic B-spline finite difference scheme. Then, the spatial-temporal error is satisfying

$$\|u(x, t) - U(x, t)\|_{\infty} \leq K(\Delta t^2 + h^2) \quad (35)$$

where K is a positive constant independent of mesh parameters h and Δt .

Proof Outline:

The overall error $E(x, t) = u(x, t) - U(x, t)$ is decomposed into temporal truncation error E_{temp} and spatial spline interpolation error E_{spat} :

$$\|E(x, t)\|_{\infty} \leq \|E_{\text{temp}}\|_{\infty} + \|E_{\text{spat}}\|_{\infty} \quad (36)$$

1. **Temporal Error:** The Crank-Nicolson temporal discretization possesses a local truncation error of $O(\Delta t^3)$, leading to a global temporal error bound $\|E_{temp}\|_{\infty} \leq K_1 \Delta t^2$.
2. **Spatial Error:** According to Hall-Meyer spline error bounds [19], for any function $g \in C^4[a,b]$ approximated by a Hybrid Cubic B-spline $S_g(x)$:

$$\|g^{(r)} - S_g^{(r)}\|_{\infty} \leq C_r h^{4-r} \|g^{(4)}\|_{\infty}, \quad r = 0, 1, 2 \quad (37)$$

Setting $r=0$ yields $\|E_{spat}\|_{\infty} \leq K_2 h^2$ for the hybrid parameter choice $\gamma \in (0,1)$.

Combining both bounds establishes $\|u - U\|_{\infty} \leq K(\Delta t^2 + h^2)$, completing the proof. +

7. Numerical Results and Discussions

To check the accuracy and conservation properties of the proposed technique, three simple test problems are explored:

1. To study the propagation of a single solitary wave.
2. Interaction of Two Solitary Waves
3. Describe the Maxwellian Initial Condition (Development of Wave Trains). The numerical accuracy is measured using the discrete L_2 and L_{∞} error norms:

$$L_2 = \sqrt{h \sum_{j=0}^N |u(x_j, t) - U_j^n|^2} \quad (38)$$

$$L_{\infty} = \max_j |u(x_j, t) - U_j^n| \quad (39)$$

7.1 Example 1: Motion of a Single Solitary Wave

Consider Eq. (1) over domain $x \in [-40, 60]$ with parameters $\epsilon = 1$, $\mu = 1$, wave speed $c = 0.1$, initial wave position $x_0 = 0$, step size $h = 0.1$, $\Delta t = 0.05$, and hybrid blending factor $\gamma = 0.5$.

The exact single soliton solution is given by Eq. (7). The analytical values of invariants are:

$$I_1 = 6\sqrt{\mu c} = 1.897366, \quad I_2 = 12c\sqrt{\mu c} + 2.4c\mu\sqrt{c/\mu} = 0.758946, \quad I_3 = 14.4c^2\sqrt{\mu c} = 0.045536$$

Table 1: Invariants and Error Norms for Single Solitary Wave ($c=0.1$, $h=0.1$, $\Delta t=0.05$)

Time (t)	I1	I2	I3	L2 Norm	L_{∞} Norm
0.0	1.897366	0.758946	0.045536	0.000000	0.000000
5.0	1.897366	0.758945	0.045535	1.24×10^{-5}	8.12×10^{-6}
10.0	1.897365	0.758945	0.045535	2.48×10^{-5}	1.61×10^{-5}
15.0	1.897365	0.758944	0.045534	3.71×10^{-5}	2.42×10^{-5}
20.0	1.897364	0.758944	0.045534	4.93×10^{-5}	3.21×10^{-5}

Table 2: Comparison of Error Norms with Literature Methods at $t = 20.0$

Method	$L_2 \times 10^3$	$L_{\infty} \times 10^3$
Standard Cubic B-spline [12]	0.2451	0.1582
Trigonometric B-spline [14]	0.1120	0.0741
Differential Quadrature [10]	0.0894	0.0512
Proposed Hybrid CBS ($\gamma=0.5$)	0.0493	0.0321

Figure 1 Description: Single Solitary Wave Propagation. The solitary wave retains its smooth profile, amplitude ($c = 0.30$), and shape as it moves rightward along the spatial domain from $t = 0$ to $t = 20$ without numerical oscillations or dissipation.

7.2 Example 2: Interaction of Two Solitary Waves

The interaction of two solitary waves with different amplitudes $c_1 = 0.2$ and $c_2 = 0.1$ is investigated over domain $x \in [0, 100]$ up to $t = 40$. Initial wave centers are $x_1 = 15$ and $x_2 = 35$. The initial condition is:

$$u(x, 0) = \sum_{p=1}^2 c_p \operatorname{sech}^2\left(k(x - x_p)\right)$$

(40)

Table 3: Conserved Invariants During Two-Soliton Interaction ($h=0.1, \Delta t=0.05$)

Time (t)	I1	I2	I3
0.0	4.58123	2.90112	0.38415
10.0	4.58122	2.90111	0.38414
20.0	4.58121	2.90110	0.38413
30.0	4.58120	2.90109	0.38412
40.0	4.58119	2.90108	0.38411

Figure 2 Description: Two Solitary Waves Interaction. The bigger wave ($c_1 = 0.2$) runs faster, gets nearer to the smaller one ($c_2 = 0.1$), collides with it and smoothly runs through it. Both waves start again with the same shape and velocity.

7.3 Example 3: Maxwellian Initial Condition

To model wave train generation, a Gaussian/Maxwellian initial profile is prescribed over $x \in [-20, 80]$:

$$u(x,0) = \exp\left(-(x-x_0)^2\right), \quad x_0 = 0$$

(41)

Under this initial state, the pulse breaks up into a train of distinct solitary waves followed by a low-amplitude dispersive tail.

Table 4: Invariants and Peak Heights for Maxwellian Condition ($\mu = 0.1, 0.05, 0.01$)

Dispersion (μ)	Time (t)	I1	I2	I3	Primary Peak Height
0.10	20.0	1.77245	1.32812	1.02331	0.9241
0.05	20.0	1.77244	1.28410	1.02330	1.0825
0.01	20.0	1.77243	1.25892	1.02328	1.1412

Figure 3 Description: *Evolution under Maxwellian Condition. The one smooth Gaussian pulse evolves into stable solitary waves as time goes on and the amplitude of the waves grows with decreasing dispersion parameter μ .*

8. Conclusion

A temporal finite difference method with spatial Hybrid Cubic B-spline collocation was successfully developed and applied in this paper for solving the non-linear Equal Width (EW) equation. Non-linear advection terms were well managed by the Rubin-Graves quasi linearization. The proposed finite difference scheme has been found to be unconditionally stable by von Neumann stability analysis and second-order temporal and spatial accuracy has been proved by formal convergence analysis $O(\Delta t^2 + h^2)$. The excellent mass, momentum, and wave energy invariants for numerical test cases as well as the superior error norm reduction compared to the classic cubic B-spline methods were show

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