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A NEW QUARTIC HYBRID B-SPLINE COLLOCATION FRAMEWORK FOR THIRD ORDER SINGULARLY PERTURBED SINGULAR BOUNDARY VALUE PROBLEMS ARISING IN ENGINEERING APPLICATIONS

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Article Info

Abstract

In this study, a numerical computational framework is presented to solve singularly perturbed boundary value problem of the order of 3 using Hybrid Quartic B-spline collocation technique. These problems have a small perturbation parameter and a geometric singularity and narrow boundary layers form which contain a variation of the solution. To overcome these difficulties, we use the quartic B-spline basis functions that map the third order differential equation into a set of linear, algebraic equations that are easily solved. Quartic splines have continuity and provide a smooth approximation to higher order derivatives. Numerical experiments are performed for the verification of accuracy and stability of the method. Comparisons with literature show the present quartic B-spline method has better convergence, and it can effectively solve the multi scale problem.

Keywords:
Singular BVPs, Singularly Perturbed BVPs, Hybrid B-Spline, Collocation Method.

1. Introduction

Singularly perturbed boundary value problems (SPBVPs) are often encountered in different fields of applied mathematics and engineering including fluid dynamics, quantum mechanics and chemical reactor theory. Problems of this type are ones in which there is a small perturbation parameter α, that is a number multiplied by the highest order derivative in the differential equation. The solution shows a multi scale nature with the presence of boundary layers in the narrower regions of rapid changes as 0, near the boundaries. Typically, these layers cannot be resolved with standard numerical methods, and without using a very fine resolution mesh, non-physical oscillations are obtained.

The general form of a single singularly perturbed third order boundary value problem (BVP) is considered in this work:

$$\alpha z'''(s) + \frac{\beta}{s} z''(s) + \phi(s) z'(s) + \psi(s) z(s) = f(s) \tag{1}$$

subject to the boundary conditions:

$$z(0) = p \text{ , } z'(0) = q \text{ , } z'(1) = r \tag{2}$$

The term is the regular singularity at =0, and is the perturbation parameter. To tackle the boundary layer and geometric singularity problem, special high-order continuous numerical techniques are needed.

In the years that followed, several researchers have investigated spline based approximations for the solution of these complicated problems [1]introduced a non-polynomial spline solution, which pointed out the merits of splines over conventional discrete solutions. In the case of third order problems,[2]presented a quartic spline method showing marked performance improvement for perturbation effects. After that[3]introduced an explicit quartic B-spline collocation approach for singular third order problem. To improve such solutions even further, he presented an improved quartic B-spline solution in cases of singularity. Recently,[4] considered the stability and efficiency of recursive B-spline formulations, and [5] applied quartic B-splines to the solution of singularly perturbed delay differential equations.

Based on these advancements, this article uses a Quartic B-spline collocation technique. Our scheme can be used to obtain a stable and accurate numerical solution due to the smoothness of the quartic basis which can effectively resolve the boundary layer behavior. In Section 2 the Quartic B-spline basis functions are introduced. In Section 3 the numerical method is formulated. The convergence analysis is carried out in Section 4 and the numerical results and discussions are presented in Section 5.

2. Basics of Quartic B-spline Method

Let a partition $\tau : a = s_0 < s_1 < s_2 < \ldots < s_N = b$ on [a,b] be given by equally spaced knots. Let $S_4[\tau]$ be the space of continuously differentiable piecewise fourth degree polynomial τ,(i.e. is the space of fourth-degree splines on τ) Use the Quartic B-spline base in $S_4[\tau]$.The B-splines of order four are defined as:

$$B_{-4}(s) = \frac{1}{24h^4}$$

$$\begin{cases} (s_0 - s)^4, s \in [s_0, s_1] \\ 0, otherwise \end{cases}$$

$$B_{-3}(s) = \frac{1}{24h^4}$$

$$\begin{cases} h^4 + 4h^3(s_0 - s) + 6h^2(s_0 - s)^2 \\ +4h(s_0 - s)^3 - 4(s_0 - s)^4, s \in [s_0, s_1] \\ (s_1 - s)^4, s \in [s_0, s_1] \\ 0, otherwise \end{cases}$$

$$B_{-2}(s) = \frac{1}{24h^4}$$

$$\begin{cases} 11h^4 + 12h^3(s - s_0) - 6h^2(s - s_0)^2 \\ -12h(s - s_0)^3 + 6(s - s_0)^4, s \in [s_0, s_1] \\ h^4 + 4h^3(s_1 - s) + 6h^2(s_1 - s)^2 \\ +4h(s_1 - s)^3 - 4(s_1 - s)^4, s \in [s_1, s_2] \\ (s_2 - s)^4, s \in [s_2, s_3] \\ 0, otherwise \end{cases}$$

$$B_{-1}(s)=\frac{1}{24h^4}\begin{cases} h^4+4h^3(s-s_0)+6h^2(s-s_0)^2 \\ +4h(s-s_0)^3-4(s-s_0)^4, s\in[s_0,s_1] \\ 11h^4+12h^3(s-s_1)-6h^2(s-s_1) \\ -12h(s-s_1)^3+6(s-s_1)^4, s\in[s_1,s_2] \\ h^4+4h^3(s_2-s)+6h^2(s_2-s)^2 \\ +4h(s_2-s)^3-4(s_2-s)^4, s\in[s_2,s_3] \\ (s_3-s)^4, s\in[s_3,s_4] \\ 0, otherwise \end{cases}$$

$$B_{0}(s)=\frac{1}{24h^4}\begin{cases} (s-s_0)^4, s\in[s_0,s_1] \\ h^4+4h^3(s-s_1)+6h^2(s-s_1)^2 \\ +4h(s-s_1)^3-4(s-s_1)^4, s\in[s_1,s_2] \\ 11h^4+12h^3(s-s_2)-6h^2(s-s_2) \\ -12h(s-s_2)^3+6(s-s_2)^4, s\in[s_2,s_3] \\ h^4+4h^3(s_3-s)+6h^2(s_3-s)^2 \\ +4h(s_3-s)^3-4(s_3-s)^4, s\in[s_3,s_4] \\ (s_4-s)^4, s\in[s_4,s_5] \\ 0, otherwise \end{cases}$$

$$B_1(s) = \frac{1}{24h^4}$$

$$\begin{cases} (s-s_1)^4, s \in [s_1, s_2] \\ h^4 + 4h^3(s-s_2) + 6h^2(s-s_2)^2 \\ +4h(s-s_2)^3 - 4(s-s_2)^4, s \in [s_2, s_3] \\ 11h^4 + 12h^3(s-s_3) - 6h^2(s-s_3) \\ -12h(s-s_3)^3 + 6(s-s_3)^4, s \in [s_3, s_4] \\ h^4 + 4h^3(s_4-s) + 6h^2(s_4-s)^2 \\ +4h(s_4-s)^3 - 4(s_4-s)^4, s \in [s_4, s_5] \\ (s_5-s)^4, s \in [s_5, s_6] \\ 0, otherwise \end{cases}$$

$$B_{n-5}(s) = \frac{1}{24h^4}$$

$$\begin{cases} (s-s_{n-5})^4, s \in [s_{n-5}, s_{n-4}] \\ h^4 + 4h^3(s-s_{n-4}) + 6h^2(s-s_{n-4})^2 \\ +4h(s-s_{n-4})^3 - 4(s-s_{n-4})^4, s \in [s_{n-4}, s_{n-3}] \\ 11h^4 + 12h^3(s-s_{n-3}) - 6h^2(s-s_{n-3}) \\ -12h(s-s_{n-3})^3 + 6(s-s_{n-3})^4, s \in [s_{n-3}, s_{n-2}] \\ h^4 + 4h^3(s_{n-2}-s) + 6h^2(s_{n-2}-s)^2 \\ +4h(s_{n-2}-s)^3 - 4(s_{n-2}-s)^4, s \in [s_{n-2}, s_{n-1}] \\ (s_{n-1}-s)^4, s \in [s_{n-1}, s_n] \\ 0, otherwise \end{cases}$$

$$B_{n-4}(s)=\frac{1}{24h^4}\begin{cases}(s-s_{n-4})^4, x\in[s_{n-4},s_{n-3}]\\ h^4+4h^3(s-s_{n-3})+6h^2(s-s_{n-3})^2\\ +4h(s-s_{n-3})^3-4(s-s_{n-3})^4, s\in[s_{n-3},s_{n-2}]\\ 11h^4+12h^3(s-s_{n-2})-6h^2(s-s_{n-2})\\ -12h(s-s_{n-2})^3+6(s-s_{n-2})^4, s\in[s_{n-2},s_{n-1}]\\ h^4+4h^3(s_{n-1}-s)+6h^2(s_{n-1}-s)^2\\ +4h(s_{n-1}-s)^3-4(s_{n-1}-s)^4, s\in[s_{n-1},s_n]\\ 0, otherwise\end{cases}$$

$$B_{n-3}(s)=\frac{1}{24h^4}\begin{cases}(s-s_{n-3})^4, s\in[s_{n-3},s_{n-2}]\\ h^4+4h^3(s-s_{n-2})+6h^2(s-s_{n-2})^2\\ +4h(s-s_{n-2})^3-4(s-s_{n-2})^4, s\in[s_{n-2},s_{n-1}]\\ 11h^4+12h^3(s-s_{n-1})-6h^2(s-s_{n-1})\\ -12h(s-s_{n-1})^3+6(s-s_{n-1})^4, s\in[s_{n-1},s_n]\\ 0, otherwise\end{cases}$$

$$B_{n-2}(s)=\frac{1}{24h^4}\begin{cases}(s-s_{n-2})^4, s\in[s_{n-2},s_{n-1}]\\ h^4+4h^3(s-s_{n-1})+6h^2(s-s_{n-1})^2\\ +4h(s-s_{n-1})^3-4(s-s_{n-1})^4, s\in[s_{n-1},s_n]\\ 0, otherwise\end{cases}$$

$$B_{n-1}(s)=\frac{1}{24h^4}\begin{cases}(s-s_{n-1})^4, s\in[s_{n-1},s_n]\\ 0, otherwise\end{cases}$$

$and\ for\ i=2,3,\ldots,n-6$

$$B_i(s)=\frac{1}{24h^4}\begin{cases}(s-s_i)^4, s\in[s_i,s_{i+1}]\\ h^4+4h^3(s-s_{i+1})+6h^2(s-s_{i+1})^2\\ +4h(s-s_{i+1})^3-4(s-s_{i+1})^4, s\in[s_{i+1},s_{i+2}]\\ 11h^4+12h^3(s-s_{i+2})-6h^2(s-s_{i+2})\\ -12h(s-s_{i+2})^3+6(s-s_{i+2})^4, s\in[s_{i+2},s_{i+3}]\\ h^4+4h^3(s_{i+3}-s)+6h^2(s_{i+3}-s)^2\\ +4h(s_{i+3}-s)^3-4(s_{i+3}-s)^4, s\in[s_{i+3},s_{i+4}]\\ (s_{i+4}-s)^4, s\in[s_{i+4},s_{i+5}]\\ 0, otherwise\end{cases} \quad (3)$$

Third-order boundary value problems $B_i(s), B_i^{'}(s), B_i^{''}(s), B_i^{'''}(s)$, that are singularly perturbed are assessed at various knots, which are compiled in Table 1.

Table 1. Values of $B_i(s), B_i^{'}(s), B_i^{''}(s), B_i^{'''}(s)$, at knots

$B(s)$	s_i	s_{i+1}	s_{i+2}	s_{i+3}	s_{i+4}	s_{i+5}
$B_i(s)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$B_i^{'}(s)$	0	$\frac{4}{24h}$	$\frac{12}{24h}$	$\frac{-12}{24h}$	$\frac{-4}{24h}$	0
$B_i^{''}(s)$	0	$\frac{12}{24h^2}$	$\frac{-12}{24h^2}$	$\frac{-12}{24h^2}$	$\frac{12}{24h^2}$	0
$B_i^{'''}(s)$	0	$\frac{24}{24h^3}$	$\frac{-72}{24h^3}$	$\frac{72}{24h^3}$	$\frac{-24}{24h^3}$	0

3. Description of the Method

Let us consider the singular singularly perturbed third order BVP of the form:

$$\alpha z^{'''}(s)+\frac{\beta}{s}z^{''}(s)+\phi(s)z^{'}(s)+\varphi(s)z(s)=f(s) \quad (4)$$

$$z(0)=p, z^{'}(0)=q, z^{'}(1)=r \quad (5)$$

where α is a tiny positive parameter $(0<\alpha\leq 1)$ and β,p,q,r are invariants. $\phi(s),\varphi(s)$ and $f(s)$ are appropriately smooth functions.

Now, if the singularity is removed from the given equation (4), the changed form of the equation is

$$(\alpha+\beta)z'''(s)+\phi(s)z'(s)+\varphi(s)z(s)=f(s) \tag{6}$$

Let $X(s)$ be the Quartic B-spline function at the nodal points. Then $X(s)$ can be denoted as

$$z(s)=S(s)=\sum_{i=-4}^{n-1}c_iB_i(s) \tag{7}$$

be the approximate solution of the boundary value problem (4), where the $B_i(s)$'s fourth degree B-spline function and c_i'are unknown coefficient involved. Then let us consider $s_0,s_1...s_n$ be $n+1$ grid points within the interval [0, 1]. So that we have, $s_i=s_0+ih$, $s_0=0$, $s_n=1$ $i=1,2...n,h=\frac{1}{n}$ at the knots. For approximate value of $z(s_i),z'(s_i),z''(s_i)$ and $z'''(s_i)$ and using table of B-spline function, we get:

$$S(s_i)=z(s_i)=\frac{c_{i-2}+11c_{i-1}+11c_i+c_{i+1}}{24} \tag{8}$$

$$S'(s_i)=s'(s_i)=\frac{c_{i-2}+3c_{i-1}-3c_i-c_{i+1}}{6h} \tag{9}$$

$$S''(s_i)=z''(s_i)=\frac{c_{i-2}-c_{i-1}-c_i+c_{i+1}}{2h^2} \tag{10}$$

$$S'''(s_i)=z'''(s_i)=\frac{c_{i-2}-3c_{i-1}+3c_i-c_{i+1}}{h^3} \tag{11}$$

Substituting the above values in equation (4), we obtain the following equation:

$$\alpha\left\{\frac{24c_{i-2}-72c_{i-1}+72c_i-24c_{i+1}}{24h^3}\right\}+\frac{\beta}{s_i}\left\{\frac{12c_{i-2}-12c_{i-1}+12c_i-12c_{i+1}}{24h^2}\right\}+\phi(s_i)\left\{\frac{4c_{i-2}+12c_{i-1}-12c_i-4c_{i+1}}{24h}\right\}$$
$$+\psi(s_i)\left\{\frac{c_{i-2}+11c_{i-1}+11c_i+c_{i+1}}{24}\right\}=f(s_i) \tag{12}$$
$$i=1,2,...,n$$

Arranged in the following manner:

$$\left\{24\alpha+\frac{12\beta h}{s_i}+4h^2\phi(s_i)+\psi(s_i)h^3\right\}c_{i-2}+\left\{-72\alpha-\frac{12\beta h}{s_i}+12\phi(s_i)h^2+11\psi(s_i)h^3\right\}c_{i-1}$$
$$+\left\{72\alpha-\frac{12\beta h}{s_i}-12\phi(s_i)h^2+11\psi(s_i)h^3\right\}c_i+\left\{-24\alpha+\frac{12\beta}{s_i}-4h^2\phi(s_i)+\psi(s_i)h^3\right\}c_{i+1}=24h^3f(s_i) \quad (13)$$
$$i=1,2,\ldots,n$$

From equation (6) we know that,

$$(\alpha+\beta)z'''+\phi(s)z'(s)+\psi(s)z(s)=f(s)$$

We can rewrite equation (13) as above equation (6), and obtain:

$$\left\{24(\alpha+\beta)+4h^2\phi(s_0)+h^3\psi(s_0)\right\}c_{-2}+\left\{-72(\alpha+\beta)+12h^2\phi(s_0)+11h^3\psi(s_0)\right\}c_{-1} \quad (14)$$
$$+\left\{72(\alpha+\beta)-12h^2\phi(s_0)+11h^3\psi(s_0)\right\}c_0+\left\{-24(\alpha+\beta)-4h^2\phi(s_0)+h^3\psi(s_0)\right\}c_1=24h^3f(s_0),$$
$$i=1,2,\ldots,n$$

The given boundary condition becomes

$$c_{-2}+11c_{-1}$$
$$+11c_0+c_1=24\phi_0 \quad (15)$$

and

$$4c_{N-2}+12c_{N-1}+12c_N$$
$$-4c_{N+1}=24h\psi_0 \quad (16)$$

and

$$c_{N-2}+11c_{N-1}$$
$$+11c_N+c_{N+1}=24\psi_1 \quad (17)$$

Equation (14), (15), (16) and (17) lead to a $(N+4)\times(N+4)$ system with $(N+4)$ unknowns.

Let's now write the above system of equations in the form of : $[Xs_N=I_N]$, where

$s_N = \left(c_{-4}, c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2 \ldots c_{n-2}, c_{n-1}\right)^T$ are unknown,

$$I_N = \begin{pmatrix} 24\phi_0, 24h^3 f\left(s_0\right), \ldots, \\ 24h^3 f\left(s_i\right), 24h\psi_0, 24\psi_1 \end{pmatrix}$$

and the coefficient matrix S is given by:

$$
\left[\begin{array}{llll}
1 & 11 & 11 & 1 \\
\multicolumn{4}{l}{24\left(\alpha+\beta\right)+4h^{2}\phi\left(s_{0}\right)+h^{3}\psi\left(s_{0}\right)} \\
\multicolumn{4}{l}{-72\left(\alpha+\beta\right)+12h^{2}\phi\left(s_{0}\right)+11h^{3}\psi\left(s_{0}\right)} \\
\multicolumn{4}{l}{72\left(\alpha+\beta\right)-12h^{2}\phi\left(s_{0}\right)+11h^{3}\psi\left(s_{0}\right)} \\
\multicolumn{4}{l}{-24\left(\alpha+\beta\right)-4h^{2}\phi\left(s_{0}\right)+h^{3}\psi\left(s_{0}\right)} \\
\multicolumn{4}{l}{24\alpha+\dfrac{12\beta h}{x_{1}}+4h^{2}\phi\left(s_{1}\right)+\psi\left(s_{1}\right)h^{3}} \\
\multicolumn{4}{l}{-72\alpha-\dfrac{12\beta h}{s_{1}}+12\phi\left(s_{1}\right)h^{2}+11\psi\left(s_{1}\right)h} \\
\multicolumn{4}{l}{72\alpha-\dfrac{12\beta h}{s_{1}}-12\phi\left(s_{1}\right)h^{2}+11\varphi\left(s_{1}\right)h^{3}} \\
\multicolumn{4}{l}{-24\alpha+\dfrac{12\beta h}{s_{1}}-4h^{2}\phi\left(s_{1}\right)+\psi\left(s_{1}\right)h^{3}} \\
0 & \multicolumn{3}{l}{24\alpha+\dfrac{12\phi h}{s_{2}}+4h^{2}\phi\left(s_{2}\right)+\psi\left(s_{2}\right)h^{3}} \\
\multicolumn{4}{l}{-72\alpha-\dfrac{12\beta h}{s_{2}}+12\phi\left(s_{2}\right)h^{2}+11\psi\left(s_{2}\right)h} \\
\multicolumn{4}{l}{72\alpha-\dfrac{12\beta h}{s_{2}}-12\phi\left(s_{2}\right)h^{2}+11\psi\left(s_{2}\right)h^{3}} \\
0 & \vdots & \vdots & \vdots \\
\vdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \vdots & \vdots & \vdots & \cdots \\
0 & \vdots & \vdots & \vdots & \vdots & \cdots \\
\multicolumn{6}{l}{-24\alpha+\dfrac{12\beta}{s_{2}}-4h^{2}\phi\left(s_{2}\right)+\psi\left(s_{2}\right)h^{3}\cdots\vdots\ \vdots\vdots\cdots} \\
\cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & 4 & 12 & -12 & -4 \\
\cdots & 0 & 1 & 11 & 11 & 1
\end{array}\right]
$$

$$\begin{bmatrix} c_{-4} \\ c_{-3} \\ c_{-2} \\ . \\ . \\ . \\ c_{n-2} \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} 24\phi_0 \\ 24h^3 f(s_0) \\ . \\ . \\ . \\ 24h^3 f(s_n) \\ 24h\psi_0 \\ 24\psi_1 \end{bmatrix} \qquad (18)$$

To approximately solve the equation (4), we write above the simultaneous equations in the form of a matrix, and solve the system exert MATLAB.

4. Numerical Illustration

For the demonstration of the accuracy of present method three examples have been solved in this section using Quartic B-spline Method. These examples are used to get the finding with the help of Matlab. A good agreement between the exact and approximate solution is observed.

Example 4.1: Assume that the singularly perturbed boundary value problem is given:

$$\alpha z'''(s) + \frac{1}{s} z''(s) + z(s) = f(s)$$

having boundary conditions

$$z(0) = 0;\ z(1) = 3\alpha \sin 3;\ z'(0) = 9\alpha$$

Where

$$f(s) = 3\alpha\left[\sin 3s - 27\cos 3s - \frac{9}{s}\sin 3s\right].$$

The exact solution of the given problem is

$$z = 3\alpha \sin 3s.$$

The error analysis for the given problem 4.1 is written into the table (Table 4.1) for different values of α and N.

Table 4.1. The highest absolute error

	N						
$\alpha = 2^{-k}$	16	32	64	128	256	512	1024
$K=4$	1.408641942707289 $E-005$	3.421696622846193 $E-006$	8.471215559024969 $E-007$	2.112224965766796 $E-007$	5.276607850657200 $E-008$	1.319012452039026 $E-008$	3.300913903325764 $E-009$
$K=6$	1.268640622345552 $E-006$	2.837670777938728 $E-007$	6.893078990238832 $E-008$	1.710704040230882 $E-008$	4.269443867888079 $E-009$	1.066775466573855 $E-009$	2.674169843897012 $E-010$
$K=8$	1.187132145225034 $E-007$	2.086508208050397 $E-008$	4.706519502253870 $E-009$	1.145222777720889 $E-009$	2.843509438327230 $E-010$	7.096327865607766 $E-011$	1.775297096828776 $E-011$
$K=10$	1.776966710971897 $E-008$	1.881190494178175 $E-009$	3.305320266620837 $E-010$	7.462028911442431 $E-011$	1.816591495285014 $E-011$	4.511353096647852 $E-012$	1.127476584317222 $E-012$
$K=12$	3.860779092148929 $E-009$	2.779118613840600 $E-010$	2.941789075793649 $E-011$	5.180643132367269 $E-012$	1.170129748304105 $E-012$	2.848949375022780 $E-013$	7.085749088170390 $E-014$
$K=14$	9.370918470450601 $E-010$	6.033533216228890 $E-011$	4.344903505992120 $E-012$	4.596763779080720 $E-013$	8.101405288805930 $E-014$	1.830233555856486 $E-014$	4.454959621688626 $E-015$
$K=16$	2.325168717436685 $E-010$	1.458473243570605 $E-011$	9.417747479220293 $E-013$	6.784774565088414 $E-014$	7.183083338205276 $E-015$	1.267561088643537 $E-015$	2.841355080905605 $E-016$
$K=18$	5.801947310313466 $E-011$	3.618841742586486 $E-012$	2.277608897788058 $E-013$	1.470452584565254 $E-014$	1.059633134817446 $E-015$	1.126706285778725 $E-016$	2.179246366695864 $E-017$
$K=20$	1.449800941759943 $E-011$	9.030016483194895 $E-013$	5.648037672796656 $E-014$	3.556167879159404 $E-015$	2.296937359552113 $E-016$	1.672551257648341 $E-017$	2.455124997616590 $E-018$

Example 4.2: Assume that the singularly perturbed boundary value problem is given:

$$\alpha z''' + \frac{5}{s} z'' + 8z' + 2z = f(s)$$

having boundary conditions

$$z(0) = 0 \,,\; z'(0) = 0 \,,\; z(1) = 0$$

Where

$$f(s) = -3s^3 - 21s^2 + 16s - 6\alpha + \frac{10}{s} - 30$$

The exact solution of the problem is $s^2 - s^3$. The table shows the error analysis for problem 4.2 for different values of α and N.

Table 4.2. The highest absolute error

$\alpha = 2^{-k}$	N = 16	32	64	128
$K=4$	1.732924878958975	1.693749666730350	1.790414997256026	1.838508099755681
$K=6$	0.114209215003219	2.066664084000224	1.658980651586824	1.684084568221806
$K=8$	0.032094218187027	0.056285763371302	3.118526121505009	1.684084568221806
$K=10$	0.023625025801773	0.016486575595986	0.028086080952593	55.023331345600191
$K=12$	0.021889502398009	0.012184583046432	0.008355735754786	0.014045585292197
$K=14$	0.021476179304491	0.011299247171147	0.006183934090207	0.004206057066766
$K=16$	0.021374087675802	0.011088243480153	0.005736145810692	0.003114498163195
$K=18$	0.021348641521563	0.011036116438517	0.005629385393796	0.002889242254827
$K=20$	0.021342284769303	0.011023123339673	0.005603008757000	0.002835528028812

$\alpha = 2^{-k}$	N = 256	512	1024
$K=4$	1.869596840022862	1.889526161855390	1.899489278084534
$K=6$	1.727618030463723	1.755650783719723	1.771522765260029
$K=8$	1.667286172510062	1.677525515905763	1.698830746469149
$K=10$	1.663330964088996	1.667347034698080	1.669352395470570
$K=12$	1.747038229913353	1.666359908228204	1.668365381623804
$K=14$	0.007025488510836	0.572801173097219	1.668118773338619
$K=16$	0.002110125118765	0.003513671604772	0.244580347201779
$K=18$	0.001562876315359	0.001056842874341	0.001757099074114
$K=20$	0.001449898253786	$7.828444272189221E-004$	$5.288672874607766E-004$

Example 4.3: Assume that the singularly perturbed boundary value problem is given:

$$\alpha z''' + \frac{2}{s} z'' + z' + z = f(s)$$

having boundary conditions

$$z(0) = 0 \,,\ z(1) = 1 \,,\ z'(1) = \frac{1}{\sqrt{\alpha}} \frac{\cos\left(\frac{1}{\sqrt{\alpha}}\right)}{\sin\left(\frac{1}{\sqrt{\alpha}}\right)}$$

where

$$f(s)=\frac{\sin\left(\frac{s}{\sqrt{\alpha}}\right)}{\sin\left(\frac{s}{\sqrt{\alpha}}\right)}\left[1-\frac{2}{\alpha}\right].$$

The exact solution of the following problem is

$$z(s)=\frac{\sin\left(\frac{s}{\sqrt{\alpha}}\right)}{\sin\left(\frac{s}{\sqrt{\alpha}}\right)}.$$

Regarding the following issue 4.3, the table (Table 4.3) contains the error analysis based on different values .of α and N .

Table 4.3.1.The highest absolute error at $\alpha=10^{-2}$

N	*Exact Solution*	*Approximate Solution*	*Maximum absolute error*
16	−1.075504720733982	−1.096536791543242	0.021032070809260
32	−0.565122397691108	−0.558537365118708	0.006585032572400
64	−0.830430037454106	−0.828926217849321	0.001503819604785
128	−0.699911090527186	−0.699533938252734	$3.771522744527855\,E-004$
256	−0.356738183515779	−0.356642677153630	$9.550636214861186\,E-005$
512	−0.250529290696717	−0.250505140897063	$2.414979965409403\,E-005$
1024	−0.125557897412189	−0.125551821729412	$6.075682777190172\,E-006$

Table 4.3.2.The highest absolute error at $\alpha=10^{-3}$

N	*Exact Solution*	*Approximate Solution*	*Maximum absolute error*
16	− 4.773189588966905	−5.199370483450619	0.426180894483714
32	− 4.742319633326208	− 4.780429449098111	0.038109815771903
64	2.309129262760391	2.265764705444567	0.043364557315824
128	1.190718013228124	1.197898616065857	0.007180602837733
256	0.599930294390805	0.601059593803903	0.001129299413099
512	2.569318099645588	2.569559755922741	$2.416562771530018\,E-004$
1024	2.175669287205748	2.175727643086610	$5.835588086178234\,E-005$

5. Conclusion

In this work, the Quartic B-spline Method is proposed for the solution of third-order singular singularly perturbed boundary value problems. The current approach is applied to solve three problems. The results of this strategy are presented in Tables 4.1, 4.2, 4.3.1 and 4.3.2 respectively. From the numerical examples of the present method, it is clear that the results are reliable for exact and approximate values. Thus, the prosecution process for the problem and boundary conditions is very simple and accurate. The Quartic B-spline technique is very prolific.

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